

OPTIMAL HEDGING UNDER FAST-VARYING STOCHASTIC VOLATILITY

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Abstract. In a market with a rough or Markovian mean-reverting stochastic volatility there is no perfect hedge. Here it is shown how various delta-type hedging strategies perform and can be evaluated in such markets. A precise characterization of the hedging cost, the replication cost caused by the volatility fluctuations, is presented in an asymptotic regime of rapid mean reversion for the volatility fluctuations. The optimal dynamic asset based hedging strategy in the considered regime is identified as the so-called “practitioners” delta hedging scheme. It is moreover shown that the performances of the delta-type hedging schemes are essentially independent of the regularity of the volatility paths in the considered regime and that the hedging costs are related to a vega risk martingale whose magnitude is proportional to a new market risk parameter.

Key words. Stochastic volatility, Rough volatility, Hedging, Risk Quantification,

AMS subject classifications. 91G80, 60H10, 60G22, 60K37.

1. Introduction. We consider an incomplete market with stochastic volatility model for the underlying. Our main objective is to characterize the performance of option hedging schemes in such markets. The rather general class of stochastic volatility models that we consider incorporates standard Markovian volatility models and also rough volatility models that have received a lot of attention recently, see [1, 15, 14, 17, 2, 10] and the literature reviews in [12, 13]. In the context of portfolio optimization Markovian models have been considered for instance in [9], while recently the non-Markovian case was considered in [4, 5, 6].

Here we model the volatility as a smooth function of a volatility factor that is a stationary Volterra type Gaussian process. In the standard volatility model the volatility factor is a mean-reverting Markov process such as an Ornstein-Uhlenbeck process. In the rough volatility model the correlation function of the volatility factor decays rapidly at the origin, faster than the decay associated with a Markov process, producing rough paths. The decay rate is characterized by the Hurst exponent H . The Gaussian volatility factor may be chosen for instance as a fractional Ornstein-Uhlenbeck process with Hurst exponent $H < 1/2$. The main asymptotic context that we consider is a rapidly mean-reverting volatility situation. The results presented here build on and extend those presented in [12] regarding option pricing for such models. Here we extend this framework to a more general class of volatility models and analyze the performance of a large class of hedging strategies for European options that we call dynamic asset (DA) based hedging schemes. A DA scheme is based on a replicating portfolio made of some number of underlyings and some amount in the bank account. In particular, this class contains the “delta”, δ , hedging strategies, in which the number of underlyings in the portfolio is the δ of the price, that is, the partial derivative of the option price with respect to the underlying price. For the classic Black-Scholes model with a constant volatility this strategy makes it possible to trade in a self-financing manner in the underlying and the bank account to perfectly replicate the payoff of the option. In the situation when the volatility is stochastic such a scheme accumulates extra cost during the lifetime of the option due to the

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fluctuations in the volatility. We consider here two main market situations: (I) the option trades at the Black-Scholes option price at the “effective volatility” or a *Black-Scholes market*, this is discussed in Section 5.1; (II) the market incorporates the effects of rapid volatility fluctuations and trades at a corrected price or a *corrected market*, this is discussed in Section 5.2. Here (I) the effective volatility refers to the root mean square of the volatility process averaged with respect to the invariant distribution of the volatility factor and (II) the corrected price refers to the Black-Scholes price at the effective volatility with a correction which follows from an asymptotic analysis of the rapidly mean-reverting situation, see Proposition 4.1. We assume that the mean reversion time of the volatility factor is small relative to the diffusion time of the underlying price. We remark that the distinction between the market situations (I) and (II) is important in the case of early exercise. Note moreover that we consider several canonical ways of computing the effective δ of the replication strategy. These are described in more detail below. In the case that “vol-of-vol” is zero, that is in the limit of small volatility fluctuations, these δ s become the standard Black-Scholes δ and the hedging strategies become the standard self-financing replicating strategy. In the case of a fluctuating volatility we present here a novel and precise characterization of the extra *hedging cost* that accumulates due to the fluctuations. For the strategy (I) this extra cost is semimartingale with in general a non-zero mean and variance that we quantify, while for the strategy (II) the extra cost is a *true martingale* and we compute its variance. We compute the costs for the DA hedging strategies and we identify the optimal hedging strategy within the DA class that minimizes the variance of the hedging cost in our regime. We allow for early exercise when evaluating the cost and we show how the cost depends on the relative exercise time. It is important to note that our results are universal in that they hold for both rough ($H < 1/2$) and classic Markovian stochastic volatility factors in the regime of rapid mean reversion. However, in a regime of slow, rather than fast mean reversion, or when $H > 1/2$, this picture changes qualitatively and results regarding these regimes will be presented elsewhere. Note, moreover, that we here consider the case with “leverage”, which means that the volatility factor is correlated with the Brownian motion driving the underlying price. In fact, in the situation with zero correlation all the hedging approaches coincide and the cost is characterized fully by the vega risk martingale.

The role of stochastic volatility for delta hedging schemes in the uncorrelated case has been discussed in [20]. Underhedged and overhedged situations are discussed there and we revisit such a characterization here in the correlated case. Superhedging schemes provide an upper bound for the replication cost [23, 22]. Here we present a statistical characterization of the hedging cost which can be used for a “value at risk” type characterization of the hedging cost. When stochastic volatility is mixing and rapidly mean-reverting the hedging cost was discussed in [21] in the case without leverage and in [7] in the case with leverage. We extend here this discussion to get explicit expressions for the hedging cost and consider more general DA hedging schemes. While we here consider hedging schemes with a view toward minimizing replication cost, portfolio construction from the point of view of utility optimization is discussed in [9] in the context of stochastic volatility in various asymptotic regimes. Our objective is indeed to characterize analytically the performance of classic (including delta) hedging schemes which plays an important role in practical risk mitigation schemes [19]. In [16] the importance of the leverage in determining risk in hedging schemes is emphasized and explored from an empirical perspective. Here we give an analytic

description of hedging risk (mean and variance of the hedging cost) in particular for the delta hedging schemes discussed in [16] in the context of leverage and rapid mean reversion.

Outline of paper: First, in Section 2 we summarize the main result of the paper. Then, in Section 3, we discuss the details of the modeling of the market with a fast mean-reverting stochastic volatility and in Section 4 we give the leading order stochastic volatility price correction for a European option in this model. Note that when we refer to “leading order” below we refer to terms of order $\sqrt{\varepsilon/T}$ or larger with ε being the mean reversion time of the volatility factor and T the time to maturity. Then we present the main result of the paper in Section 5 on the characterization of the hedging costs for the various hedging schemes that we consider. We discuss in more detail the main effective parameters in Sections 6 and 7. We specialize to the case of a call option in Section 8 and we present numerical illustrations. We finally provide some concluding remarks in Section 9.

2. Summary of Main Results. We consider in this section hedging of a European option with payoff $h(X_T)$ with T the maturity and X_t the underlying. The underlying is assumed to follow a diffusion process with a stochastic volatility as described in Section 3, Eq. (3.1). In this paper we do not consider short rate effects, corresponding to assuming as numeraire the zero coupon bond with maturity T . Moreover, we do not consider effects associated with dividends, market price of volatility risk or transaction cost. An important assumption is, however, that we assume a non-zero “leverage”, which means that the volatility factor is driven by a Brownian motion that is correlated with the Brownian motion driving the underlying, see Eq. (3.6) below. Our main objective is to identify analytically the *hedging cost*. We assume a regime where the mean reversion time of the volatility factor is small relative to the diffusion time of the underlying which is on the scale of the maturity T , that is, we consider a *rapidly mean-reverting stationary* volatility. We present asymptotic results in the regime of rapid mean reversion and below we make precise the sense of the approximation. Our class of volatility models incorporates standard Markovian volatility models and rough volatility models.

Let the root mean square or “historical” volatility be denoted by $\bar{\sigma}$. Moreover, let $Q^{(0)}(t, x; \sigma)$ be the standard Black-Scholes (European option) price at volatility level σ evaluated at time t and current value x for the underlying. Then the price that incorporates the leading order correction due to the rapidly mean-reverting stochastic volatility is:

$$P(t, x) = Q^{(0)}(t, x; \bar{\sigma}) + D(T - t)(x\partial_x(x^2\partial_x^2))Q^{(0)}(t, x; \bar{\sigma}), \quad (2.1)$$

see Section 4. Here D is an *effective pricing parameter* that can be calibrated from observations of the implied volatility skew, see Section 6.

We construct a replicating portfolio so that a_t is the number of underlyings at time t and b_t is the amount in the bank account. The value of the portfolio is then

$$V_t = a_t X_t + b_t. \quad (2.2)$$

The portfolio is required to replicate the price of the option so it replicates the payoff at maturity $V_T = h(X_T)$. The net payment stream provided by the market over the time interval $(0, T)$ due to changes in the price of the underlying is

$$\int_0^T a_s dX_s.$$

The change in the portfolio value that is not “financed” by the market has to be paid by the portfolio holder and we call this the cost function:

$$E_T = h(X_T) - \int_0^T a_s dX_s.$$

This hedging scheme is called a DA scheme if a_t is a function of t and X_t . The general class of DA hedging schemes contains the delta hedging strategies, that is to say, the strategies in which the number $a_t = \delta(t, X_t)$ of underlyings in the portfolio at time t is the derivative of the price of the option with respect to the value of the underlying. We consider first two main delta hedging strategies characterized by the chosen “delta”:

(HW): The delta of the corrected price:

$$\delta^{\text{HW}}(t, x) = \partial_x P(t, x), \quad (2.3)$$

with P given by (2.1).

(BS): The delta of the Black-Scholes price at the implied volatility:

$$\delta^{\text{BS}}(t, x) = \partial_x Q^{(0)}(t, x; \sigma)|_{\sigma=\sigma(t, x)}, \quad (2.4)$$

with the implied volatility $\sigma(t, x)$ solving

$$P(t, x) = Q^{(0)}(t, x; \sigma(t, x)). \quad (2.5)$$

In the case that the volatility is constant and equal to $\bar{\sigma}$, corresponding to the standard Black-Scholes model, these approaches coincide and the portfolios are self-financing. In the case that the volatility is fluctuating, the model is incomplete and we accumulate additional hedging cost during the lifetime of the option. We remark that with no leverage effect (which means that the volatility factor is independent of the Brownian motion driving the underlying price), then $D = 0$ and the two approaches coincide and give the same hedging cost.

By (2.5), the delta of hedging approach (HW) corresponds to

$$\delta^{\text{HW}}(t, x) = \partial_x Q^{(0)}(t, x; \sigma(x, t)) + \partial_\sigma Q^{(0)}(t, x; \sigma(x, t)) \times \partial_x \sigma(x, t).$$

This scheme is referred to as the *minimum variance delta* in the recent paper [16] by Hull and White. They find by empirical comparison of a few strategies that this hedging approach is the one associated with minimum hedging risk or cost variance. In [16] the minimum variance delta and enhanced performance is motivated by the presence of leverage. Here we quantify the means and variances of the hedging costs analytically and correspondingly identify analytically the hedging approach with minimum hedging cost variance in our setting, which is not the (HW) scheme.

The costs of the hedging strategies are characterized by the three market parameters

$$\bar{\sigma}, \quad D, \quad \Gamma,$$

see Section 6. The first and second are sufficient to characterize the price as we have remarked above, the third is a *hedging risk* parameter. Consider the situation when we construct a hedging portfolio of value $P(t, X_t)$ and write the total hedging cost at maturity T by

$$E_T^C = P(0, X_0) + Y_T^C, \quad C = \text{HW, BS}, \quad (2.6)$$

for the two choices of hedging delta. Here X_0 is the underlying value at initiation time $t = 0$ and $P(0, X_0)$ the initiation cost of the portfolio. Then in a sense made precise below the random part of the cost at maturity Y_T^{HW} is

$$Y_T^{\text{HW}} = \Gamma \int_0^T (x^2 \partial_x^2)^2 Q^{(0)}(s, X_s; \bar{\sigma}) dB_s = \Gamma \int_0^T \frac{\partial_\sigma Q^{(0)}(s, X_s; \bar{\sigma})}{T-s} dB_s,$$

for B a standard Brownian motion. If the price sensitivity to volatility changes, the vega, is small, then the vega risk is small as well. The sensitivity to vega in the cost accumulation becomes larger as one approaches maturity. The cost does not depend on the market pricing parameter D , and hence it does not depend on the leverage correlation parameter ρ either (ρ is the correlation between the volatility factor and the Brownian motion driving the underlying price, see Eq. (3.6) below). However, it is proportional to the hedging risk parameter Γ which does not depend on ρ and which is the central new parameter. Thus, the hedging approach is *leverage compensating* in that it “immunizes” the portfolio with respect to “leverage risk”. In the particular case of a European call option with strike K , i.e. $h(x) = (x - K)^+$, we have $\mathbb{E}[Y_T^{\text{HW}} | \mathcal{F}_0] = 0$ and

$$\text{Var}(Y_T^{\text{HW}} | \mathcal{F}_0) = \left(\frac{K\Gamma}{\bar{\sigma}}\right)^2 \frac{1}{2\pi} \int_0^1 \exp\left(-\frac{d_-^2}{1+s}\right) \frac{1}{\sqrt{1-s^2}} ds, \quad (2.7)$$

with the standard Black-Scholes parameter

$$d_- = \frac{\log(X_0/K)}{\sqrt{\tau}} - \frac{\sqrt{\tau}}{2}, \quad \tau = \bar{\sigma}^2 T. \quad (2.8)$$

Here the expectation and variance are taken conditionally on the information at time zero. We show this hedging cost variance at maturity in Figure 2.1 as a function of relative time to maturity, $\tau = \bar{\sigma}^2 T$, and moneyness, $m = X_0/K$.

We next state the important result that leverage makes the “practitioners” hedging approach superior. We have explicitly $\mathbb{E}[Y_T^{\text{BS}} | \mathcal{F}_0] = 0$ and

$$\text{Var}(Y_T^{\text{BS}} | \mathcal{F}_0) = \text{Var}(Y_T^{\text{HW}} | \mathcal{F}_0) \left(1 - \left(\frac{D}{\bar{\sigma}\Gamma}\right)^2\right), \quad \text{with } \left|\frac{D}{\bar{\sigma}\Gamma}\right| \leq |\rho| \leq 1, \quad (2.9)$$

which implies $\text{Var}(Y_T^{\text{BS}} | \mathcal{F}_0) \leq \text{Var}(Y_T^{\text{HW}} | \mathcal{F}_0)$. The main result of this paper is then set forth in Section 8.3, Proposition 8.3: *the (BS) hedging scheme minimizes the hedging cost variance among all DA hedging schemes*, thus is the true minimum variance hedging scheme in the regime discussed here!

In Section 7 we discuss the explicit expressions of the effective market parameters when the volatility model is the exponential of a standard or fractional (with Hurst exponent $H < 1/2$) Ornstein-Uhlenbeck process. In this case we have

$$\frac{D}{\bar{\sigma}\Gamma} \approx \rho. \quad (2.10)$$

Note that the implementation of the delta hedging schemes (HW) and (BS) requires the knowledge of the two effective market parameters $\bar{\sigma}$ and D . Below we will also discuss the case when we choose a “homogenized” or “historical” delta:

(H): The delta of the Black-Scholes price at the historical volatility:

$$\delta^{\text{H}}(t, x) = \partial_x Q^{(0)}(t, x; \bar{\sigma}). \quad (2.11)$$

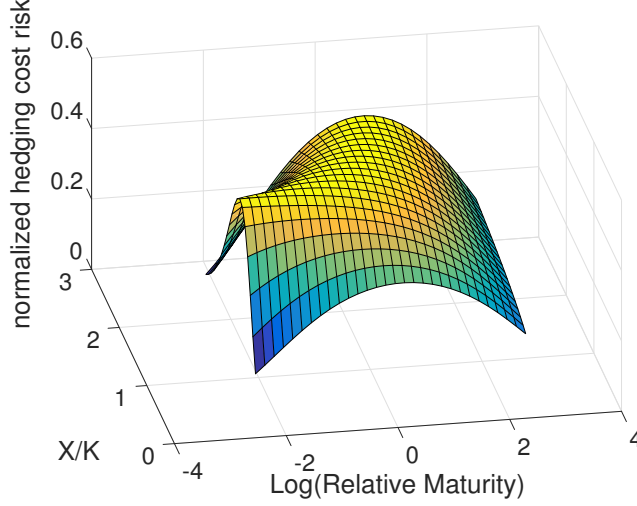


FIG. 2.1. The figure shows the normalized hedging cost standard deviation $St.Dev(Y_T^{HW})\bar{\sigma}/(KT)$ as function of relative time to maturity $\tau = \bar{\sigma}^2 T$ and moneyness $m = X_0/K$.

This hedging scheme (H) can be implemented with *only the knowledge of $\bar{\sigma}$* and does not require calibration based on pricing data. However, in all cases implementing the hedging scheme and simultaneously characterizing the hedging cost mean and variance requires the knowledge of all three market parameters $(\bar{\sigma}, D, \Gamma)$. For the scheme (H) we can write as in Eq. (2.6) for the hedging cost at maturity:

$$E_T^H = P(0, X_0) + Y_T^H, \quad (2.12)$$

and it follows from Proposition 8.3 that $\text{Var}(Y_T^H | \mathcal{F}_0) \geq \text{Var}(Y_T^{BS} | \mathcal{F}_0)$. In particular for a European call with strike K we have $\mathbb{E}[Y_T^H | \mathcal{F}_0] = 0$ and

$$\text{Var}(Y_T^H | \mathcal{F}_0) = \text{Var}(Y_T^{HW} | \mathcal{F}_0) + \left(\frac{KD}{\bar{\sigma}^2}\right)^2 \hat{w}^H(d_-), \quad (2.13)$$

with d_- given by (2.8) and

$$\hat{w}^H(d) = \frac{2}{\pi} \int_0^1 \exp\left(-\frac{d^2}{1+s}\right) \frac{1}{\sqrt{1-s^2}} \left[d^4 \frac{(1-s)^2}{(1+s)^4} + d^2 \frac{6s(1-s)}{(1+s)^3} + \frac{3s^2}{(1+s)^2} - \frac{1}{2} \right] ds - \frac{d^2 \exp(-d^2)}{2\pi}. \quad (2.14)$$

Below we will also present the results for the hedging costs in the case with early exercise $t < T$. Before we present such hedging risk characterizations in cases with general payoffs and exercise times in Section 5 we discuss the modeling of the stochastic volatility in Section 3 and the asymptotic pricing formula in Section 4.

3. A Class of Fast Mean Reverting Rough Volatility Models. The price of the risky asset follows the stochastic differential equation:

$$dX_t = \sigma_t^\varepsilon X_t dW_t^*, \quad (3.1)$$

with W_t^* a standard Brownian motion. The stochastic volatility is

$$\sigma_t^\varepsilon = F(Z_t^\varepsilon), \quad (3.2)$$

and is a stationary process. The volatility is not in general a Gaussian process and is driven by the volatility factor Z_t^ε that is a scaled stationary Gaussian process:

$$Z_t^\varepsilon = \sigma_z \int_{-\infty}^t \mathcal{K}^\varepsilon(t-s) dW_s, \quad (3.3)$$

with W_t a standard Brownian motion and where

$$\mathcal{K}^\varepsilon(t) = \frac{1}{\sqrt{\varepsilon}} \mathcal{K}\left(\frac{t}{\varepsilon}\right). \quad (3.4)$$

We have introduced the mean reversion time scale ε which will be the small time scale in our problem. It means in particular that we consider contracts whose time to maturity is long compared to the natural time scale of the volatility factor. Thus, we refer to the volatility factor and associated volatility process as *rapidly mean-reverting*.

We make the following assumption regarding the volatility model:

- (i) $\mathcal{K} \in L^2(0, \infty)$ with $\int_0^\infty \mathcal{K}^2(u) du = 1$ and $\mathcal{K} \in L^1(0, \infty)$.
- (ii) There is a $d > 1$ so that:

$$|\mathcal{K}(t)| = O(t^{-d}) \quad \text{as } t \rightarrow \infty. \quad (3.5)$$

- (iii) F is smooth increasing and bounded from below (away from zero) and from above.

Under these conditions Z_t^ε has mean zero and variance σ_z^2 . We assume that W_t^* is a Brownian motion that is correlated to the stochastic volatility through

$$W_t^* = \rho W_t + \sqrt{1 - \rho^2} W_t', \quad (3.6)$$

where the Brownian motion W_t' is independent of W_t . The function F is assumed to be one-to-one, positive-valued, smooth, bounded and with bounded derivatives. Accordingly, the filtration \mathcal{F}_t generated by (W_t', W_t) is also the one generated by X_t . Indeed, it is equivalent to the one generated by (W_t^*, W_t) , or (W_t^*, Z_t^ε) . Since F is one-to-one, it is equivalent to the one generated by $(W_t^*, \sigma_t^\varepsilon)$. Since F is positive-valued, it is equivalent to the one generated by $(W_t^*, (\sigma_t^\varepsilon)^2)$, or X_t .

The volatility may thus be a mixing process or a rough process with rapid decay of correlations at the origin. In the latter case the volatility is neither a martingale nor a Markov process. We discuss next some particular volatility models.

3.1. Standard Ornstein-Uhlenbeck Model. Here we discuss the standard model where Z_t^ε is the scaled Ornstein Uhlenbeck (OU) process. It has the form (3.3-3.4) with $\mathcal{K}(t) = \sqrt{2} \exp(-t)$. The OU process Z_t^ε is a centered Gaussian process with covariance of the form

$$\mathbb{E}[Z_t^\varepsilon Z_{t+s}^\varepsilon] = \sigma_z^2 \mathcal{C}_Z\left(\frac{s}{\varepsilon}\right), \quad (3.7)$$

with $\mathcal{C}_Z(s) = \exp(-|s|)$. It solves a Langevin equation driven by standard Brownian motion. It is a martingale and a Markov process, which allows for the use of stochastic calculus [7].

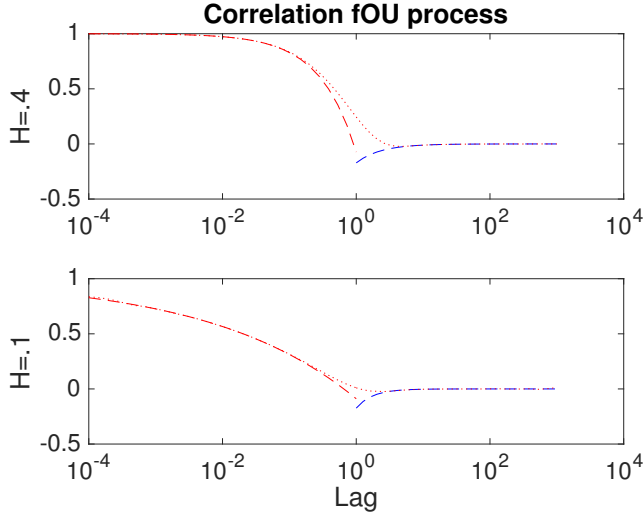


FIG. 3.1. The dotted lines show the correlation functions C_Z for $H = .4$ (top plot) and $H = .1$ (bottom plot). Note that for large lags the correlation function is slightly negative. Note also the rapid decay of the correlations at the origin, increasingly so with smaller H . The dashed lines are the approximations in Eqs. (3.9) and (3.10).

3.2. Rough Volatility Models. We discuss here the model where Z_t^ε is the scaled fractional Ornstein Uhlenbeck (fOU) process with Hurst exponent $H \in (0, 1/2)$. This process is described in more detail in Appendix B, it has the form (3.3-3.4) with

$$\mathcal{K}(t) = \frac{\sqrt{2 \sin(\pi H)}}{\Gamma(H + \frac{1}{2})} \left[t^{H-\frac{1}{2}} - \int_0^t (t-s)^{H-\frac{1}{2}} e^{-s} ds \right]. \quad (3.8)$$

The fOU process Z_t^ε is a centered Gaussian process with covariance of the form (3.7) with $C_Z(0) = 1$, see Eq. (B.6). Compared to the standard OU process addressed in the previous subsection, we allow here for more general volatility factors to capture the situations discussed in a number of recent empirical findings that the volatility process is rough corresponding to rapid decay of C_Z at the origin [14]. We arrive at such a situation by assuming that the OU process is driven by a fractional Brownian motion with Hurst exponent $H \in (0, 1/2)$ rather than a standard Brownian motion [3]. As described in Appendix B this gives a volatility factor that is *rough*. We have specifically now that the covariance function C_Z is rough at zero in the sense:

$$C_Z(s) = 1 - \frac{1}{\Gamma(2H + 1)} s^{2H} + o(s^{2H}), \quad s \ll 1, \quad (3.9)$$

while it is integrable and it decays as s^{2H-2} at infinity:

$$C_Z(s) = \frac{1}{\Gamma(2H - 1)} s^{2H-2} + o(s^{2H-2}), \quad s \gg 1, \quad (3.10)$$

see Figure 3.1. This behavior of the covariance function is inherited by the volatility process σ_t^ε itself, see Eqs. (B.13) and (B.14). For more details regarding this model we refer to [13].

4. Prices of European Options. We are interested in computing the option price defined as the martingale

$$M_t = \mathbb{E}[h(X_T) \mid \mathcal{F}_t], \quad (4.1)$$

where h is a smooth function, $0 \leq t \leq T$, and the price follows the equation (3.1) with the volatility model described in Section 3. In fact weaker assumptions are possible for h , as we only need to control the function $Q_t^{(0)}(x)$ defined below rather than h , as is discussed in [11, Section 4]. We here assume that the interest rate and market price of risk are zero.

We introduce the standard Black-Scholes operator at zero interest rate and constant volatility σ :

$$\mathcal{L}_{\text{BS}}(\sigma) = \partial_t + \frac{1}{2}\sigma^2 x^2 \partial_x^2. \quad (4.2)$$

We exploit the fact that the price process is a martingale to obtain an approximation, via constructing an explicit function $P(t, x)$ so that $P(T, x) = h(x)$ and so that $P(t, X_t)$ is a martingale up to first order corrective terms in ε . Then, indeed $P(t, X_t)$ gives the approximation for M_t up to first order in ε . The leading order price is the price at the homogenized or constant parameters. The following proposition gives the first-order correction to the expression for the martingale M_t in the regime of ε small.

PROPOSITION 4.1. *We have*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1/2} \sup_{t \in [0, T]} \mathbb{E} [|M_t - P(t, X_t)|^2]^{1/2} = 0, \quad (4.3)$$

where

$$P(t, x) = Q_t^{(0)}(x) + \sqrt{\varepsilon} \rho Q_t^{(1)}(x), \quad (4.4)$$

$Q_t^{(0)}(x)$ is deterministic and given by the Black-Scholes formula with constant volatility $\bar{\sigma}$,

$$\mathcal{L}_{\text{BS}}(\bar{\sigma})Q_t^{(0)}(x) = 0, \quad Q_T^{(0)}(x) = h(x), \quad (4.5)$$

with

$$\bar{\sigma}^2 = \langle F^2 \rangle = \int_{\mathbb{R}} F(\sigma_z z)^2 p(z) dz, \quad (4.6)$$

$p(z)$ the pdf of the standard normal distribution, $Q_t^{(1)}(x)$ is the deterministic correction solving

$$\mathcal{L}_{\text{BS}}(\bar{\sigma})Q_t^{(1)}(x) = -\bar{D}(x \partial_x (x^2 \partial_x^2))Q_t^{(0)}(x), \quad Q_T^{(1)}(x) = 0. \quad (4.7)$$

The deterministic correction is

$$Q_t^{(1)}(x) = (T - t) \bar{D}(x \partial_x (x^2 \partial_x^2))Q_t^{(0)}(x), \quad (4.8)$$

where the coefficient \bar{D} is defined by

$$\bar{D} = \sigma_z \int_0^\infty \left[\iint_{\mathbb{R}^2} F(\sigma_z z) (FF')(\sigma_z z') p_{\mathcal{C}_{\mathcal{K}}(s, 0)}(z, z') dz dz' \right] \mathcal{K}(s) ds, \quad (4.9)$$

with $p_C(z, z')$ the pdf of the bivariate normal distribution with mean zero and covariance matrix

$$\begin{pmatrix} 1 & C \\ C & 1 \end{pmatrix}, \quad (4.10)$$

and

$$C_{\mathcal{K}}(s, s') = \int_0^\infty \mathcal{K}(s+v)\mathcal{K}(s'+v)dv. \quad (4.11)$$

The mixing (Markov) case is addressed in [7, 8] and the rough case is derived in [13]. The above statement concerns a generalization of the volatility model in [13] and can be derived via a straightforward modification of the proof presented there. Thus, we see that the effect of the volatility fluctuations gives a price modification that is of the order of $\varepsilon^{1/2}$ and which is determined by the effective parameter \bar{D} only. The main result of this paper is a precise statistical characterization of hedging cost in the context of fast mean-reverting stochastic volatility. Our novel analysis uses the analytic framework set forth in [13]. As for the case of option prices the hedging cost results are for the general volatility model (3.2). Therefore they apply in particular in a uniform way to the cases of Markov and rough volatility.

It is important to note, however, that the ‘long-memory’ case addressed in [12], corresponding to $H > 1/2$, is different. In this case the volatility ‘history’ plays a crucial role and gives a qualitatively different picture from the point of view of pricing and hedging. This is also the case for small volatility fluctuations as presented in [11]. These cases will be discussed elsewhere.

5. Hedging Cost Accumulation. In the following sections we derive the results for the cost associated with the hedging approaches introduced above in the context of European options. We summarize in the next proposition these results. We introduced the hedging approaches (H), (HW), (BS) in Section 2. In Section 5.4 we introduce the modified approach (\tilde{H}) where the delta is chosen to be δ^H as in the (H) approach, however, the value of the portfolio is chosen to be $P(t, x)$ rather than $Q_t^{(0)}(x)$ as in the (H) approach. The following proposition follows directly from Propositions 5.3, 5.4, 5.6, 5.8 and Section 5.4. It gives the leading-order expressions of the expectations and the variances of the hedging costs (the leading order is $\sqrt{\varepsilon}$ for the expectation and ε for the variance).

PROPOSITION 5.1. *If we write the hedging cost in the form*

$$E_t^C = P(0, X_0) + Y_t^C, \quad \text{for } C = \text{H, HW, BS, } \tilde{H}, \quad (5.1)$$

then we have

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\left(\varepsilon^{-1/2} \mathbb{E}[Y_t^H | \mathcal{F}_0] - \frac{(t-T)\rho\bar{D}}{T\bar{\sigma}^2} g(X_0, T) \right)^2 \right]^{1/2} = 0, \quad (5.2)$$

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\left(\varepsilon^{-1/2} \mathbb{E}[Y_t^C | \mathcal{F}_0] \right)^2 \right]^{1/2} = 0, \quad \text{for } C = \text{HW, BS, } \tilde{H}, \quad (5.3)$$

and

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\left| \varepsilon^{-1} \text{Var}(Y_t^{\text{HW}} | \mathcal{F}_0) - \frac{\bar{\Gamma}^2}{\bar{\sigma}^2} v(X_0, t, T) \right| \right] = 0, \quad (5.4)$$

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\left| \varepsilon^{-1} \text{Var}(Y_t^{\text{C}} | \mathcal{F}_0) - \frac{\bar{\Gamma}^2}{\bar{\sigma}^2} v(X_0, t, T) - \frac{\rho^2 \bar{D}^2}{\bar{\sigma}^4} w^{\text{C}}(X_0, t, T) \right| \right] = 0, \quad (5.5)$$

for $\text{C} = \text{H}, \text{BS}, \tilde{\text{H}}$, where \bar{D} and $\bar{\Gamma}$ are the parameters given by (4.9) and (5.41) and g, v, w^{C} are cost mean and variance functions that depend on the payoff function h .

The explicit forms of g, v, w^{C} are given in Section 8, Proposition 8.1, in the case of European call options $h(x) = (x - K)^+$.

Remark. In the following we show that, up to terms of order $o(\sqrt{\varepsilon})$:

$$\begin{aligned} E_t^{\text{HW}} - P(0, X_0) &= N_t^{(1)}, \\ E_t^{\text{BS}} - P(0, X_0) &= N_t^{(1)} + \sqrt{\varepsilon} \rho N_t^{(2)}, \\ E_t^{\tilde{\text{H}}} - P(0, X_0) &= N_t^{(1)} + \sqrt{\varepsilon} \rho \tilde{N}_t^{(2)}, \end{aligned}$$

where $N^{(1)}$, resp. $N^{(2)}, \tilde{N}^{(2)}$, are the martingales defined in Eq. (5.13), resp. Eq. (5.14), Eq. (5.57). It is in fact the negative correlation between $N^{(1)}$ and $N^{(2)}$ that makes the (BS) approach superior, see Section 8.3. In the case of the approach (H) the hedging cost is characterized by

$$E_t^{\text{H}} - Q_0^{(0)}(X_0) = N_t^{(1)} + \sqrt{\varepsilon} \rho \bar{D} \int_0^t (x \partial_x (x^2 \partial_x^2)) Q_s^{(0)}(X_s) ds,$$

with

$$\mathbb{E} \left[\sqrt{\varepsilon} \rho \bar{D} \int_0^t (x \partial_x (x^2 \partial_x^2)) Q_s^{(0)}(X_s) ds \mid \mathcal{F}_0 \right] = \left(\frac{t}{T} \right) \left(P(0, X_0) - Q_0^{(0)}(X_0) \right).$$

Here and below $(x \partial_x (x^2 \partial_x^2)) Q_s^{(0)}(X_s)$ stands for $(x \partial_x (x^2 \partial_x^2)) Q_s^{(0)}(x)$ evaluated at $x = X_s$. We next derive these results.

5.1. Hedging Cost Process with (H) Hedging Strategy. Consider the (H) hedging scheme. We assume that the *effective volatility* $\bar{\sigma}$ is known and choose here the number of underlyings in the replicating portfolio as the “ δ ” of the Black-Scholes price evaluated at the *effective volatility* and the current price for the underlying. Thus, we consider here the situation with “homogenized” or “historical” delta:

$$a_t^{\text{H}} = \delta^{\text{H}}(t, X_t), \quad \delta^{\text{H}}(t, x) = \partial_x Q_t^{(0)}(x), \quad (5.6)$$

as in Eq. (2.11). Moreover, in this section we choose the value of the portfolio V_t^{H} to replicate the Black-Scholes price $Q_t^{(0)}(X_t)$ evaluated at the effective volatility:

$$V_t^{\text{H}} = Q_t^{(0)}(X_t), \quad 0 \leq t \leq T, \quad (5.7)$$

and $b_t^{\text{H}} = Q_t^{(0)}(X_t) - a_t^{\text{H}} X_t$. As mentioned this hedging scheme can then be implemented knowing only $\bar{\sigma}$. As we will show though in order to characterize the hedging cost mean and variance we need to know also the effective market parameters (D, Γ) .

The portfolio replicates the payoff at maturity $V_T^H = Q_T^{(0)}(X_T) = h(X_T)$. The cost function is:

$$E_t^H = V_t^H - \int_0^t a_s^H dX_s, \quad (5.8)$$

with in particular $E_0^H = Q_0^{(0)}(X_0)$. We aim to understand how this cost can be characterized.

Using the fact that $Q^{(0)}$ solves the Black-Scholes equation we find

$$\begin{aligned} dE_t^H &= dV_t^H - a_t^H dX_t = \left(\partial_t + \frac{1}{2}(\sigma_t^\varepsilon)^2 (x^2 \partial_x^2) \right) Q_t^{(0)}(X_t) dt + \partial_x Q_t^{(0)}(X_t) dX_t - a_t^H dX_t \\ &= \frac{1}{2} \left((\sigma_t^\varepsilon)^2 - \bar{\sigma}^2 \right) (x^2 \partial_x^2) Q_t^{(0)}(X_t) dt. \end{aligned} \quad (5.9)$$

We remark that we can write

$$dE_t^H = \frac{1}{2} \left((\sigma_t^\varepsilon)^2 - \bar{\sigma}^2 \right) \frac{\nu_t(X_t)}{\bar{\sigma}(T-t)} dt,$$

where we introduced the ‘‘vega’’:

$$\nu_t(x) = \partial_{\bar{\sigma}} Q_t^{(0)}(x) = \bar{\sigma}(T-t) (x^2 \partial_x^2) Q_t^{(0)}(x). \quad (5.10)$$

Note that in the special case of constant volatility we have $\sigma_t^\varepsilon \equiv \bar{\sigma}$ and thus $dE_t^H = 0$, which means that the cost is deterministic and given by the Black-Scholes price:

$$\mathbb{E}[E_t^H | \mathcal{F}_0] = Q_0^{(0)}(X_0), \quad \text{Var}(E_t^H | \mathcal{F}_0) = 0, \quad 0 \leq t \leq T.$$

In the rapid stochastic volatility case (3.2), we can identify the leading-order terms of the cost. Two equivalent expressions can be determined as shown in Lemma 5.2. They will be useful to compute the mean and variance of the cost in the next propositions.

LEMMA 5.2. *The hedging cost satisfies*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1/2} \sup_{t \in [0, T]} \mathbb{E} \left[|E_t^H - \hat{E}_t^H|^2 \right]^{1/2} = 0, \quad (5.11)$$

where

$$\hat{E}_t^H = Q_0^{(0)}(X_0) + \varepsilon^{1/2} \rho (Q_0^{(1)}(X_0) - Q_t^{(1)}(X_t)) + N_t^{(1)} + \varepsilon^{1/2} \rho N_t^{(2)}, \quad (5.12)$$

$N_t^{(1)}$ and $N_t^{(2)}$ are the martingales starting at zero

$$N_t^{(1)} = \int_0^t (x^2 \partial_x^2) Q_s^{(0)}(X_s) d\psi_s^\varepsilon, \quad (5.13)$$

$$N_t^{(2)} = \int_0^t (x \partial_x) Q_s^{(1)}(X_s) \sigma_s^\varepsilon dW_s^*, \quad (5.14)$$

with

$$\psi_t^\varepsilon = \mathbb{E} \left[\frac{1}{2} \int_0^T \left((\sigma_s^\varepsilon)^2 - \bar{\sigma}^2 \right) ds \mid \mathcal{F}_t \right]. \quad (5.15)$$

We also have

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1/2} \sup_{t \in [0, T]} \mathbb{E} [|E_t^H - \check{E}_t^H|^2]^{1/2} = 0, \quad (5.16)$$

where

$$\check{E}_t^H = Q_0^{(0)}(X_0) + \varepsilon^{1/2} \rho \bar{D} \int_0^t (x \partial_x (x^2 \partial_x^2)) Q_s^{(0)}(X_s) ds + N_t^{(1)}. \quad (5.17)$$

Note that the difference in Eq (5.12) can be interpreted as the cost of trading the correction over the interval $(0, t)$ and $N_t^{(2)}$ is (minus) the martingale part of this cost which gives Eq. (5.17) in view of the problems solved by $Q^{(0)}$ and $Q^{(1)}$ as stated in Proposition 4.1. Moreover, we can write from (4.8):

$$\lim_{\Delta t \downarrow 0} \frac{\mathbb{E} [\check{E}_{t+\Delta t}^H - \check{E}_t^H | \mathcal{F}_t]}{\Delta t} = \frac{\varepsilon^{1/2} \rho Q_t^{(1)}(X_t)}{T - t},$$

so that the current ‘‘coherent cost flux’’ corresponds to the accumulation of the cost of the correction over the interval remaining until maturity.

Proof. Let ϕ_t^ε be defined as the expected accumulated square volatility deviation in between the present and maturity:

$$\phi_t^\varepsilon = \mathbb{E} \left[\frac{1}{2} \int_t^T ((\sigma_s^\varepsilon)^2 - \bar{\sigma}^2) ds \mid \mathcal{F}_t \right]. \quad (5.18)$$

Then we have

$$\phi_t^\varepsilon = \psi_t^\varepsilon - \frac{1}{2} \int_0^t ((\sigma_s^\varepsilon)^2 - \bar{\sigma}^2) ds,$$

where the martingale ψ_t^ε is defined by (5.15). $(\psi_t^\varepsilon)_{t \in [0, T]}$ is a square-integrable martingale that satisfies the following properties:

- The quadratic covariation of ψ^ε and W is

$$d \langle \psi^\varepsilon, W \rangle_t = \vartheta_t^\varepsilon dt, \quad \vartheta_t^\varepsilon = \sigma_z \int_t^T \mathbb{E} [FF'(Z_s^\varepsilon) \mid \mathcal{F}_t] \mathcal{K}^\varepsilon(s - t) ds, \quad (5.19)$$

with \mathcal{K}^ε of the form (3.4).

- There exists a constant K_T such that we have almost surely

$$\sup_{t \in [0, T]} |\vartheta_t^\varepsilon| \leq K_T \varepsilon^{1/2}. \quad (5.20)$$

The first part was proved in [12, Lemma B.1]. The second part follows from the fact that $\mathcal{K}^\varepsilon(t) = \mathcal{K}(t/\varepsilon)/\sqrt{\varepsilon}$, $\mathcal{K} \in L^1(0, \infty)$.

We define the martingales starting from zero at time zero:

$$dN_t^{(0)} = (x \partial_x) Q_t^{(0)}(X_t) \sigma_t^\varepsilon dW_t^*, \quad (5.21)$$

$$dN_t^{(3)} = (x \partial_x (x^2 \partial_x^2)) Q_t^{(0)}(X_t) \sigma_t^\varepsilon \phi_t^\varepsilon dW_t^*. \quad (5.22)$$

Then Eqs. (31) and (36) in [13] read:

$$\begin{aligned}
\frac{1}{2}((\sigma_t^\varepsilon)^2 - \bar{\sigma}^2)(x^2 \partial_x^2) Q_t^{(0)}(X_t) dt &= dQ_t^{(0)}(X_t) - dN_t^{(0)}, \quad (5.23) \\
dQ_t^{(0)}(X_t) &= -d[\phi_t^\varepsilon(x^2 \partial_x^2) Q_t^{(0)}(X_t) + \varepsilon^{1/2} \rho Q_t^{(1)}(X_t)] \\
&\quad + \frac{1}{2}(x^2 \partial_x^2(x^2 \partial_x^2)) Q_t^{(0)}(X_t)((\sigma_t^\varepsilon)^2 - \bar{\sigma}^2) \phi_t^\varepsilon dt \\
&\quad + \frac{\varepsilon^{1/2}}{2} \rho(x^2 \partial_x^2) Q_t^{(1)}(X_t)((\sigma_t^\varepsilon)^2 - \bar{\sigma}^2) dt \\
&\quad + \rho(x \partial_x(x^2 \partial_x^2)) Q_t^{(0)}(X_t)(\sigma_t^\varepsilon \vartheta_t^\varepsilon - \varepsilon^{1/2} \bar{D}) dt \\
&\quad + dN_t^{(0)} + dN_t^{(1)} + \varepsilon^{1/2} \rho dN_t^{(2)} + dN_t^{(3)}. \quad (5.24)
\end{aligned}$$

In [13] it is shown that the third, fourth, and fifth terms of the right-hand side of (5.24) are smaller than $\varepsilon^{1/2}$. That is, if we introduce for any $t \in [0, T]$:

$$R_{t,T}^{(1)} = \int_t^T \frac{1}{2}(x^2 \partial_x^2(x^2 \partial_x^2)) Q_s^{(0)}(X_s)((\sigma_s^\varepsilon)^2 - \bar{\sigma}^2) \phi_s^\varepsilon ds, \quad (5.25)$$

$$R_{t,T}^{(2)} = \int_t^T \frac{\varepsilon^{1/2}}{2} \rho(x^2 \partial_x^2) Q_s^{(1)}(X_s)((\sigma_s^\varepsilon)^2 - \bar{\sigma}^2) ds, \quad (5.26)$$

$$R_{t,T}^{(3)} = \int_t^T \rho(x \partial_x(x^2 \partial_x^2)) Q_s^{(0)}(X_s)(\sigma_s^\varepsilon \vartheta_s^\varepsilon - \varepsilon^{1/2} \bar{D}) ds, \quad (5.27)$$

we have for $j = 1, 2, 3$,

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1/2} \sup_{t \in [0, T]} \mathbb{E}[(R_{t,T}^{(j)})^2]^{1/2} = 0. \quad (5.28)$$

From Proposition 4.1 we have that

$$\begin{aligned}
-dQ_t^{(1)}(X_t) &= \bar{D}(x \partial_x(x^2 \partial_x^2)) Q_t^{(0)}(X_t) dt - dN_t^{(2)} \\
&\quad - \frac{1}{2}(x^2 \partial_x^2) Q_t^{(1)}(X_t)((\sigma_t^\varepsilon)^2 - \bar{\sigma}^2) dt. \quad (5.29)
\end{aligned}$$

It then follows from (5.9)-(5.23)-(5.24)-(5.29) that

$$\begin{aligned}
dE_t^H &= \varepsilon^{1/2} \rho \bar{D}(x \partial_x(x^2 \partial_x^2)) Q_t^{(0)}(X_t) dt + dR_{t,T}^{(1)} + dR_{t,T}^{(3)} \\
&\quad - d[\phi_t^\varepsilon(x^2 \partial_x^2) Q_t^{(0)}(X_t)] + dN_t^{(1)} + dN_t^{(3)}. \quad (5.30)
\end{aligned}$$

It follows from Lemma A.2 that the first term in the second line of Eq. (5.30) is small:

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1/2} \sup_{t \in [0, T]} \mathbb{E} \left[\left| \int_0^t -d[\phi_s^\varepsilon(x^2 \partial_x^2) Q_s^{(0)}(X_s)] \right|^2 \right]^{1/2} = 0, \quad (5.31)$$

and the third term, i.e. the martingale $N_t^{(3)}$, is small as well:

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1/2} \sup_{t \in [0, T]} \mathbb{E} \left[|N_t^{(3)}|^2 \right]^{1/2} = 0.$$

We then get (5.16-5.17). Finally we remark from (5.26) and (5.29) that

$$\check{E}_t^H = \hat{E}_t^H + R_{0,T}^{(2)} - R_{t,T}^{(2)},$$

so that (5.28) gives (5.11-5.12). \square

We next consider the expected hedging cost. We find that, if we exercise at some time $0 \leq t \leq T$, the extra hedging cost beyond the Black-Scholes price at the effective volatility is the fraction t/T of the price correction at the initiation time:

PROPOSITION 5.3. *The mean hedging cost satisfies*

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\left(\varepsilon^{-1/2} \mathbb{E}[E_t^H - E_0^H \mid \mathcal{F}_0] - \frac{t}{T} \rho Q_0^{(1)}(X_0) \right)^2 \right]^{1/2} = 0, \quad (5.32)$$

with $E_0^H = Q_0^{(0)}(X_0)$.

Therefore, we have

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\left(\varepsilon^{-1/2} (\mathbb{E}[E_t^H \mid \mathcal{F}_0] - P(0, X_0)) - \frac{t-T}{T} \rho Q_0^{(1)}(X_0) \right)^2 \right]^{1/2} = 0, \quad (5.33)$$

which gives (5.2).

Proof. From (5.17) we have

$$\varepsilon^{-1/2} \mathbb{E}[\check{E}_t^H - Q_0^{(0)}(X_0) \mid \mathcal{F}_0] = \rho \bar{D} \int_0^t \mathbb{E}[(x \partial_x (x^2 \partial_x^2)) Q_s^{(0)}(X_s) \mid \mathcal{F}_0] ds. \quad (5.34)$$

Using (5.16), Lemma A.11 (Eq. (A.17)), and dominated convergence theorem, it follows that

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\left(\varepsilon^{-1/2} \mathbb{E}[E_t^H - E_0^H \mid \mathcal{F}_0] - \rho \bar{D} \int_0^t \mathbb{E}[(x \partial_x (x^2 \partial_x^2)) Q_s^{(0)}(\tilde{X}_s) \mid \mathcal{F}_0] ds \right)^2 \right] = 0, \quad (5.35)$$

with

$$d\tilde{X}_t = \bar{\sigma} \tilde{X}_t dW_t^*, \quad \tilde{X}_0 = X_0. \quad (5.36)$$

On the one hand, from (4.7) we get

$$\begin{aligned} \rho \bar{D} \int_0^t \mathbb{E}[(x \partial_x (x^2 \partial_x^2)) Q_s^{(0)}(\tilde{X}_s) \mid \mathcal{F}_0] ds &= -\rho \mathbb{E} \left[\int_0^t \mathcal{L}_{\text{BS}}(\bar{\sigma}) Q_s^{(1)}(\tilde{X}_s) ds \mid \mathcal{F}_0 \right] \\ &= -\rho \mathbb{E}[Q_t^{(1)}(\tilde{X}_t) - Q_0^{(1)}(\tilde{X}_0) \mid \mathcal{F}_0], \end{aligned}$$

which is equal to 0 at $t = 0$ and equal to $\rho Q_0^{(1)}(X_0)$ at $t = T$.

On the other hand, we have by Itô's formula and (4.5) that

$$\mathbb{E} \left[(x \partial_x (x^2 \partial_x^2)) Q_s^{(0)}(\tilde{X}_s) \mid \mathcal{F}_0 \right] = (x \partial_x (x^2 \partial_x^2)) Q_0^{(0)}(X_0),$$

which shows that the integral term in (5.35) is a linear function in t . Therefore it is equal to $(t/T) \rho Q_0^{(1)}(X_0)$, which completes the proof of (5.32). \square

We are also interested in the risk or uncertainty in the hedging cost if we exercise at or before expiry. We find that the magnitude of the cost fluctuations is of order $\sqrt{\varepsilon}$. We have an explicit integral expression for the variance of the hedging cost fluctuations (to leading order ε) as explained in the following proposition:

PROPOSITION 5.4. *The asymptotic variance of the cost fluctuations satisfies*

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\varepsilon^{-1} \text{Var}(E_t^H - E_0^H \mid \mathcal{F}_0) - \mathcal{V}_t^{(1)}(X_0) - 2\mathcal{V}_t^{(2)}(X_0) - \mathcal{V}_t^{(3)}(X_0) \right] = 0, \quad (5.37)$$

with

$$\begin{aligned} \mathcal{V}_t^{(1)}(x_0) &= 2\rho^2 \bar{D}^2 \int_{\mathbb{R}} dz p(z) \int_0^t ds (t-s) \left((x\partial_x(x^2\partial_x^2))Q_s^{(0)}(x_0 e^{\bar{\sigma}\sqrt{sz} - \bar{\sigma}^2 s/2}) \right)^2 \\ &\quad - \left(\frac{t}{\bar{T}} \rho Q_0^{(1)}(x_0) \right)^2, \end{aligned} \quad (5.38)$$

$$\begin{aligned} \mathcal{V}_t^{(2)}(x_0) &= \rho^2 \bar{D}^2 \int_{\mathbb{R}} dz p(z) \int_0^t ds (t-s) \left(((x\partial_x)^2(x^2\partial_x^2))Q_s^{(0)}(x_0 e^{\bar{\sigma}\sqrt{sz} - \bar{\sigma}^2 s/2}) \right) \\ &\quad \times \left((x^2\partial_x^2)Q_s^{(0)}(x_0 e^{\bar{\sigma}\sqrt{sz} - \bar{\sigma}^2 s/2}) \right), \end{aligned} \quad (5.39)$$

$$\mathcal{V}_t^{(3)}(x_0) = \bar{\Gamma}^2 \int_{\mathbb{R}} dz p(z) \int_0^t ds \left((x^2\partial_x^2)Q_s^{(0)}(x_0 e^{\bar{\sigma}\sqrt{sz} - \bar{\sigma}^2 s/2}) \right)^2. \quad (5.40)$$

Here $p(z)$ is the pdf of the standard normal distribution, $\bar{\Gamma}$ is the parameter

$$\bar{\Gamma}^2 = 2\sigma_z^2 \int_0^\infty \int_s^\infty \left[\iint_{\mathbb{R}^2} FF'(\sigma_z z) FF'(\sigma_z z') p_{C_{\mathcal{K}}(s,s')}(z, z') dz dz' \right] \mathcal{K}(s) \mathcal{K}(s') ds' ds, \quad (5.41)$$

and p_C is the pdf of the bivariate normal distribution with covariance matrix (4.10) and $C_{\mathcal{K}}(s, s')$ is defined by (4.11).

Proof. From (5.11) and (5.17), we can write

$$\varepsilon^{-1} \text{Var}(E_t^H - E_0^H \mid \mathcal{F}_0) = V_1^\varepsilon + 2V_2^\varepsilon + V_3^\varepsilon + o(1), \quad (5.42)$$

$$V_1^\varepsilon = \text{Var} \left(\rho \bar{D} \int_0^t (x\partial_x(x^2\partial_x^2))Q_s^{(0)}(X_s) ds \mid \mathcal{F}_0 \right), \quad (5.43)$$

$$V_2^\varepsilon = \varepsilon^{-1/2} \text{Cov} \left(\rho \bar{D} \int_0^t (x\partial_x(x^2\partial_x^2))Q_s^{(0)}(X_s) ds, N_t^{(1)} \mid \mathcal{F}_0 \right), \quad (5.44)$$

$$V_3^\varepsilon = \varepsilon^{-1} \text{Var}(N_t^{(1)} \mid \mathcal{F}_0). \quad (5.45)$$

Note that we have $x^2\partial_x^2 = (x\partial_x)^2 - x\partial_x$. It follows that

$$\mathcal{L}_{\text{BS}}(\bar{\sigma}) \left((x\partial_x)^j (x^2\partial_x^2) \right) Q_t^{(0)}(x) = 0, \quad j = 0, 1, \dots$$

Then one can show that V_1^ε converges in L^1 to $\mathcal{V}_t^{(1)}(X_0)$ (given by Eq. (5.38)) by Lemma A.13-Eq. (A.21) and Proposition 5.3. Similarly, using the expression (5.13) of $N^{(1)}$, one can show that V_2^ε and V_3^ε converge in L^1 to $\mathcal{V}_t^{(2)}(X_0)$ and $\mathcal{V}_t^{(3)}(X_0)$ (given by Eqs. (5.39) and (5.40)) by Lemma A.2 and by Lemma A.13-Eqs. (A.22-A.23) respectively. \square

We illustrate the above result in the case of a European call option in Section 8.

5.2. Hedging Cost Process using (HW) Hedging Strategy. In this section we analyze the situation when we use the hedging scheme (HW) described by Eq. (2.3) where we use a “corrected delta” to construct the portfolio. That is, we now use the corrected Black-Scholes price in Proposition 4.1 and associated delta and value function.

Thus, we construct a replicating portfolio so that a_t^{HW} is the number of underlyings at time t and b_t^{HW} the amount in the bank account according to the corrected strategy. The value of the portfolio is now

$$V_t^{\text{HW}} = a_t^{\text{HW}} X_t + b_t^{\text{HW}}, \quad (5.46)$$

and we choose

$$a_t^{\text{HW}} = \delta^{\text{HW}}(t, X_t), \quad \delta^{\text{HW}}(t, x) = \partial_x P(t, x) = \partial_x (Q_t^{(0)} + \varepsilon^{1/2} \rho Q_t^{(1)})(x). \quad (5.47)$$

We moreover require the portfolio to replicate the corrected option price so that the value of the portfolio is

$$V_t^{\text{HW}} = P(t, X_t), \quad 0 \leq t \leq T, \quad (5.48)$$

and $b_t^{\text{HW}} = P(t, X_t) - a_t^{\text{HW}} X_t$. Again the portfolio replicates the payoff at maturity $V_T^{\text{HW}} = P(T, X_T) = h(X_T)$. The financing cost of the portfolio is

$$E_t^{\text{HW}} = V_t^{\text{HW}} - \int_0^t a_s^{\text{HW}} dX_s, \quad (5.49)$$

with in particular $E_0^{\text{HW}} = P(0, X_0)$. We aim to understand how the cost is affected by using the corrected strategy. The following lemma shows that, by using the corrected hedging strategy, we have in the incomplete market *restored the situation with existence of a self-financing replicating portfolio to the order of the approximation in the mean*. Moreover the hedging cost is characterized by the martingale $N^{(1)}$ defined by (5.13).

LEMMA 5.5. *The cost of the corrected hedging strategy satisfies*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1/2} \sup_{t \in [0, T]} \mathbb{E} \left[\left(E_t^{\text{HW}} - P(0, X_0) - N_t^{(1)} \right)^2 \right]^{1/2} = 0, \quad (5.50)$$

where $N_t^{(1)}$ is the martingale defined in Lemma 5.2, Eq. (5.13).

Proof. In view of Eqs. (4.5) and (4.7) we find

$$\begin{aligned} dE_t^{\text{HW}} &= dV_t^{\text{HW}} - a_t^{\text{HW}} dX_t \\ &= \left(\partial_t + \frac{1}{2} (\sigma_t^\varepsilon)^2 x^2 \partial_x^2 \right) P(t, X_t) dt + \partial_x P(t, X_t) dX_t - a_t^{\text{HW}} dX_t \\ &= \frac{1}{2} \left((\sigma_t^\varepsilon)^2 - \bar{\sigma}^2 \right) (x^2 \partial_x^2) P(t, X_t) dt - \varepsilon^{1/2} \rho \bar{D} (x \partial_x (x^2 \partial_x^2)) Q_t^{(0)}(X_t) dt. \end{aligned}$$

We define \tilde{E}^{HW} by

$$d\tilde{E}_t^{\text{HW}} = \frac{1}{2} \left((\sigma_t^\varepsilon)^2 - \bar{\sigma}^2 \right) (x^2 \partial_x^2) Q_t^{(0)}(X_t) dt - \varepsilon^{1/2} \rho \bar{D} (x \partial_x (x^2 \partial_x^2)) Q_t^{(0)}(X_t) dt, \quad (5.51)$$

starting from $\tilde{E}_0^{\text{HW}} = P(0, X_0)$. Therefore

$$E_t^{\text{HW}} - \tilde{E}_t^{\text{HW}} = \rho \varepsilon^{1/2} \int_0^t \frac{1}{2} \left((\sigma_s^\varepsilon)^2 - \bar{\sigma}^2 \right) (x^2 \partial_x^2) Q_s^{(1)}(X_s) ds,$$

and we get from Lemma A.12:

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1/2} \sup_{t \in [0, T]} \mathbb{E}[(E_t^{\text{HW}} - \tilde{E}_t^{\text{HW}})^2]^{1/2} = 0. \quad (5.52)$$

We have from (5.9) and (5.51):

$$d\tilde{E}_t^{\text{HW}} - dE_t^{\text{H}} = -\varepsilon^{1/2} \rho \bar{D}(x \partial_x (x^2 \partial_x^2)) Q_t^{(0)}(X_t) dt.$$

Using (5.17) we get

$$\begin{aligned} \tilde{E}_t^{\text{HW}} - E_t^{\text{H}} + \check{E}_t^{\text{H}} &= \tilde{E}_0^{\text{HW}} - E_0^{\text{H}} + \check{E}_0^{\text{H}} + N_t^{(1)} \\ &= P(0, X_0) - Q_0^{(0)}(X_0) + Q_0^{(0)}(X_0) + N_t^{(1)} = P(0, X_0) + N_t^{(1)}. \end{aligned}$$

Using (5.16) we find that

$$\begin{aligned} &\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1/2} \sup_{t \in [0, T]} \mathbb{E} \left[\left(\tilde{E}_t^{\text{HW}} - P(0, X_0) - N_t^{(1)} \right)^2 \right]^{1/2} \\ &= \lim_{\varepsilon \rightarrow 0} \varepsilon^{-1/2} \sup_{t \in [0, T]} \mathbb{E}[(E_t^{\text{H}} - \check{E}_t^{\text{H}})^2]^{1/2} = 0, \end{aligned}$$

which gives the desired result with Eq. (5.52). \square

This lemma allows us to characterize the mean and variance of the cost of the corrected hedging strategy.

PROPOSITION 5.6. *The mean extra hedging cost beyond the corrected price is zero:*

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\left(\varepsilon^{-1/2} \mathbb{E}[E_t^{\text{HW}} - E_0^{\text{HW}} \mid \mathcal{F}_0] \right)^2 \right]^{1/2} = 0, \quad (5.53)$$

with $E_0^{\text{HW}} = P(0, X_0)$. The variance of the cost fluctuations satisfies

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\left[\varepsilon^{-1} \text{Var}(E_t^{\text{HW}} - E_0^{\text{HW}} \mid \mathcal{F}_0) - \mathcal{V}_t^{(3)}(X_0) \right] \right] = 0, \quad (5.54)$$

where $\mathcal{V}_t^{(3)}$ is given by (5.40).

Proof. The result on the mean follows from Lemma 5.5 and the fact that $N_t^{(1)}$ is a zero-mean martingale. The result on the variance follows from Lemma 5.5 and the formula for the asymptotic variance of $N_t^{(1)}$ obtained in Proposition 5.4. \square

5.3. Hedging Cost with (BS) Hedging Strategy. Consider here the hedging approach (BS) described in Section 2, that is using the delta δ^{BS} and as value function P . Here $Q^{(j)}(t, x; \sigma)$, $j = 0, 1$ stands for $Q_t^{(j)}(x)$ with the constant volatility σ instead of $\bar{\sigma}$. Using a similar technique as in the derivation of Lemma 5.2 and Proposition 5.4 we find

LEMMA 5.7. *The cost for the hedging scheme (BS), E^{BS} , satisfies*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1/2} \sup_{t \in [0, T]} \mathbb{E} \left[(E_t^{\text{BS}} - \hat{E}_t^{\text{BS}})^2 \right]^{1/2} = 0, \quad (5.55)$$

where

$$\hat{E}_t^{\text{BS}} = P(0, X_0) + N_t^{(1)} + \varepsilon^{1/2} \rho \tilde{N}_t, \quad (5.56)$$

$N_t^{(1)}$ is the martingale defined by (5.13), and \tilde{N}_t is the martingale defined by

$$\tilde{N}_t = \bar{D} \int_0^t \tilde{\mathcal{H}}_s(X_s) \sigma_s^\varepsilon dW_s^*, \quad (5.57)$$

$$\tilde{\mathcal{H}}_s(x) = \frac{1}{\bar{D}} \left((x \partial_x) - \left(\frac{x \partial_x \partial_\sigma Q^{(0)}(s, x; \bar{\sigma})}{\partial_\sigma Q^{(0)}(s, x; \bar{\sigma})} \right) \right) Q^{(1)}(s, x; \bar{\sigma}). \quad (5.58)$$

Proof. The implied volatility $\sigma(t, x)$ is such that

$$Q^{(0)}(t, x; \sigma(t, x)) = P(t, x) = Q^{(0)}(t, x; \bar{\sigma}) + \sqrt{\varepsilon} \rho Q^{(1)}(t, x; \bar{\sigma}).$$

so that we have:

$$\sigma(t, x) - \bar{\sigma} = \frac{\sqrt{\varepsilon} \rho Q^{(1)}(t, x; \bar{\sigma})}{\partial_\sigma Q^{(0)}(t, x; \bar{\sigma})} + o(\sqrt{\varepsilon}).$$

The (BS) delta is:

$$\begin{aligned} \delta^{\text{BS}}(t, x) &= \left(\partial_x Q^{(0)}(t, x; \sigma) \right) \Big|_{\sigma=\sigma(t, x)} \\ &= \left(\partial_x (Q^{(0)}(t, x; \bar{\sigma}) + \partial_\sigma Q^{(0)}(t, x; \bar{\sigma})(\sigma - \bar{\sigma})) \right) \Big|_{\sigma=\sigma(t, x)} + o(\sqrt{\varepsilon}), \end{aligned}$$

so that we can write:

$$\delta^{\text{BS}}(t, x) = \delta^{\text{H}}(t, x) + \sqrt{\varepsilon} \rho Q^{(1)}(t, x; \bar{\sigma}) \left(\frac{\partial_x \partial_\sigma Q^{(0)}(t, x; \bar{\sigma})}{\partial_\sigma Q^{(0)}(t, x; \bar{\sigma})} \right) + o(\sqrt{\varepsilon}).$$

Then it follows from Eqs. (5.8) and (5.12) that the cost is

$$\begin{aligned} E_t^{\text{BS}} &= P(t, X_t) - \int_0^t \delta^{\text{BS}}(s, X_s) ds \\ &= E_t^{\text{H}} + \sqrt{\varepsilon} \rho Q_t^{(1)}(X_t) - \sqrt{\varepsilon} \rho \int_0^t \left(\frac{x \partial_x \partial_\sigma Q^{(0)}(s, X_s; \bar{\sigma})}{\partial_\sigma Q^{(0)}(s, X_s; \bar{\sigma})} \right) Q^{(1)}(s, X_s; \bar{\sigma}) \sigma_s^\varepsilon dW_s^* \\ &= P(0, X_0) + N_t^{(1)} + \sqrt{\varepsilon} \rho N_t^{(2)} \\ &\quad - \sqrt{\varepsilon} \rho \int_0^t \left(\frac{x \partial_x \partial_\sigma Q^{(0)}(s, X_s; \bar{\sigma})}{\partial_\sigma Q^{(0)}(s, X_s; \bar{\sigma})} \right) Q^{(1)}(s, X_s; \bar{\sigma}) \sigma_s^\varepsilon dW_s^* + o(\sqrt{\varepsilon}) \\ &= P(0, X_0) + N_t^{(1)} + \sqrt{\varepsilon} \rho \tilde{N}_t + o(\sqrt{\varepsilon}), \end{aligned}$$

with \tilde{N}_t defined by (5.57). \square

This lemma allows us to characterize the mean and variance of the cost of the (BS) hedging scheme.

PROPOSITION 5.8. *The mean and variance of the cost fluctuations satisfy*

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\left(\varepsilon^{-1/2} \mathbb{E} [E_t^{\text{BS}} - E_0^{\text{BS}} | \mathcal{F}_0] \right)^2 \right]^{1/2} = 0, \quad (5.59)$$

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\left[\varepsilon^{-1} \text{Var} (E_t^{\text{BS}} - E_0^{\text{BS}} | \mathcal{F}_0) - \tilde{\mathcal{V}}_t^{(1)}(X_0) - 2\tilde{\mathcal{V}}_t^{(2)}(X_0) - \tilde{\mathcal{V}}_t^{(3)}(X_0) \right] \right] = 0, \quad (5.60)$$

with $E_0^{\text{BS}} = P(0, X_0)$,

$$\tilde{\mathcal{V}}_t^{(1)}(x_0) = \rho^2 \bar{D}^2 \bar{\sigma}^2 \int_{\mathbb{R}} dz p(z) \int_0^t ds \left(\tilde{\mathcal{H}}_s(x_0 e^{\bar{\sigma} \sqrt{s} z - \bar{\sigma}^2 s/2}) \right)^2, \quad (5.61)$$

$$\begin{aligned} \tilde{\mathcal{V}}_t^{(2)}(x_0) &= \rho^2 \bar{D}^2 \int_{\mathbb{R}} dz p(z) \int_0^t ds \tilde{\mathcal{H}}_s(x_0 e^{\bar{\sigma} \sqrt{s} z - \bar{\sigma}^2 s/2}) \\ &\quad \times \left(x^2 \partial_x^2 Q_s^{(0)}(x_0 e^{\bar{\sigma} \sqrt{s} z - \bar{\sigma}^2 s/2}) \right), \end{aligned} \quad (5.62)$$

$$\tilde{\mathcal{V}}_t^{(3)}(x_0) = \bar{\Gamma}^2 \int_{\mathbb{R}} dz p(z) \int_0^t ds \left(x^2 \partial_x^2 Q_s^{(0)}(x_0 e^{\bar{\sigma} \sqrt{s} z - \bar{\sigma}^2 s/2}) \right)^2, \quad (5.63)$$

where $\bar{\Gamma}$ is defined by (5.41) and $\tilde{\mathcal{H}}_s(x)$ is defined by (5.58).

5.4. Hedging Cost with a Modified (H) Hedging Strategy. To facilitate comparison of the schemes at early exercise times we here consider the hedging scheme (H) using the delta at the Black-Scholes price at the effective volatility, δ^{H} , however, modified in that the portfolio value is chosen to be the corrected price $P(t, x)$ rather than the price $Q_t^{(0)}(x)$ at the effective volatility. We label this approach ($\tilde{\text{H}}$).

Note that using Eq. (5.12) we can write that the accumulated asymptotic hedging cost until time t has the form:

$$E_t^{\tilde{\text{H}}} = P(t, X_t) - \int_0^t \delta^{\text{H}}(s, X_s) dX_s = P(0, X_0) + N_t^{(1)} + \varepsilon^{1/2} \rho N_t^{(2)} + o(\varepsilon^{1/2}). \quad (5.64)$$

We then find that the hedging cost is characterized by Lemma 5.7 and Proposition 5.8 upon the replacements:

$$\tilde{N} \mapsto N^{(2)}, \quad \bar{D} \tilde{\mathcal{H}}_t(x) \mapsto (x \partial_x) Q_t^{(1)}(x).$$

6. On Estimation of Effective Market Parameters. For the above results to be useful we must be able to estimate the three market parameters discussed in Section 2

$$\bar{\sigma}, \quad D = \sqrt{\varepsilon} \rho \bar{D}, \quad \Gamma = \sqrt{\varepsilon} \bar{\Gamma}. \quad (6.1)$$

We refer to $D = \sqrt{\varepsilon} \rho \bar{D}$ as an *effective pricing parameter* with the price correction being scaled by this parameter. The effective pricing parameter can together with the effective or historical volatility, $\bar{\sigma}$, be calibrated from observation of vanilla option prices and the associated implied volatility skew, see for instance the discussions in [7, 8].

The parameter $\Gamma = \sqrt{\varepsilon} \bar{\Gamma}$ is a *hedging risk parameter* and the magnitude of vega risk martingale $N^{(1)}$ scales with this parameter. The hedging cost parameter can be calibrated from historical data. Indeed, by constructing the (HW) hedge for instance and recording the accumulated cost over times $t_i, i = 0, \dots, n$ say, we will have an estimate of the martingale $N^{(1)}$ at these times from which the parameter $\sqrt{\varepsilon} \bar{\Gamma}$ can be estimated via a least squares procedure that fits the empirical variance of the martingale $N^{(1)}$ with the formula (5.45)-(5.40) in which only $\varepsilon \bar{\Gamma}^2$ is unknown. Then this ‘‘historical’’ hedging risk parameter estimate can be used to project future hedging cost (mean and variance), thus, the theory provides a bridge from historical to future hedging cost.

We remark also that in more complex market situations and modeling, incorporating for instance (random) market price of volatility risk and interest rate, there will be additional parameters to estimate.

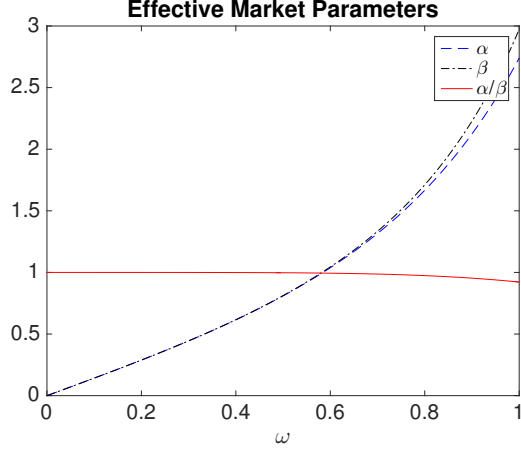


FIG. 7.1. Market parameters α , β in ExpOU case.

7. Effective Market Parameters Deriving from ExpfOU. We discuss here the exponential fractional Ornstein-Uhlenbeck process or ExpfOU model. We then define the volatility by $\sigma_t^\varepsilon = F(Z_t^\varepsilon)$ with

$$F(z) = \bar{\sigma} \exp\left(\frac{\omega z}{\sigma_z} - \omega^2\right),$$

which is such that $\langle F^2 \rangle = \bar{\sigma}^2$. Here, $\omega > 0$ is a fluctuation parameter that measures the typical amplitude of the relative fluctuations of the volatility:

$$\frac{\langle F^4 \rangle - \langle F^2 \rangle^2}{\langle F^2 \rangle^2} = e^{4\omega^2} - 1.$$

We introduce two parameters that summarize the information contained in \mathcal{K} as defined in (3.4) (and the function $\mathcal{C}_\mathcal{K}$ defined in terms of \mathcal{K} by (4.11)):

$$\alpha = \frac{\bar{D}}{\bar{\sigma}^3} = \omega e^{-\frac{\omega^2}{2}} \int_0^\infty e^{2\omega^2 \mathcal{C}_\mathcal{K}(s,0)} \mathcal{K}(s) ds, \quad (7.1)$$

$$\beta = \frac{\bar{\Gamma}}{\bar{\sigma}^2} = \left(\omega^2 \int_0^\infty \int_0^\infty e^{4\omega^2 \mathcal{C}_\mathcal{K}(s,s')} \mathcal{K}(s) \mathcal{K}(s') ds ds' \right)^{1/2}. \quad (7.2)$$

These two parameters (with $\bar{\sigma}$) are necessary and sufficient to compute the corrected price and hedging cost. In the case of a “classic” ExpOU model with $\mathcal{K}(t) = \sqrt{2} \exp(-t)$ they are given explicitly by:

$$\alpha = e^{-\omega^2/2} \frac{e^{2\omega^2} - 1}{\sqrt{2}\omega}, \quad \beta = \sqrt{\frac{1}{2} E_1(4\omega^2) - \frac{\gamma}{2} - \ln(2\omega)},$$

with $E_1(z) = \int_z^\infty \frac{e^{-t}}{t} dt$ the exponential integral function and $\gamma \simeq 0.577$ the Euler constant. We plot α and β as function of ω in the ExpOU case in Figure 7.1. Note that $\alpha/\beta \leq 1$ is nearly independent of ω and approximately equal to 1 for $\omega \leq 1$.

8. Hedging Cost Statistics for European Call Options.

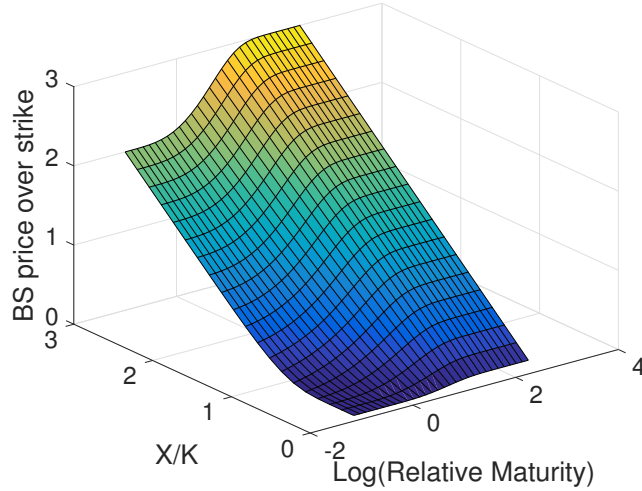


FIG. 8.1. The figure shows the European call option price relative to strike: $Q_0^{(0)}/K$. It is plotted as a function of Log relative maturity, $\log_{10}(\tau) = \log_{10}(T\bar{\sigma}^2)$, and moneyness, $m = X_0/K$. For short maturities we see the call payoff while there is a transition regime to the large maturity limit, the identity, for relative maturity roughly around unity.

8.1. Price and its Delta and their Corrected Versions. In Figure 8.2 we show the normalized hedging cost correction $\bar{\sigma}^2 Q_0^{(1)}/(K\bar{D})$ and in Figure 8.1 we show the Black-Scholes price relative to strike $Q_0^{(0)}/K$ for comparison. Note that for small maturities and moneyness the mean correction is relatively more important. In Figure 8.3 we show the delta for the Black-Scholes price and in Figure 8.4 we show the delta for the normalized price correction. Note that for short maturities and around the money the Black-Scholes delta at the effective volatility gives an *overhedged* situation in that the delta associated with the correction is negative. We also see that for short maturities and moneyness the Black-Scholes delta gives an *underhedged* situation in that the delta associated with the correction is positive.

8.2. Call Hedging Risk. In Proposition 5.1 we gave the expressions of the means and variances of the hedging costs in the case of a general payoff. The explicit expressions for the normalized functions g, v, w^C , for $C = H, BS, \tilde{H}$, follow from the propositions in the Section 5. Here we consider the situation with a European call. Then we can use the results in Appendix C, Eqs. (C.1-C.8), to get explicit expressions for the normalized functions g, v, w^C .

PROPOSITION 8.1. In the case of a European call option $h(x) = (x - K)^+$ and using the notation in Proposition 5.1, the normalized functions g, v, w^C depend on

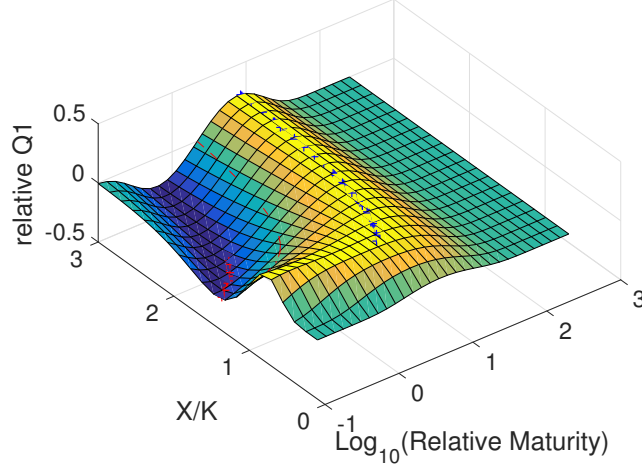


FIG. 8.2. The figure shows the mean normalized hedging cost correction for the European call option: $\bar{\sigma}^2 Q_0^{(1)} / (K\bar{D}) = -d_- \exp(-d_-^2/2) / \sqrt{2\pi}$. It is plotted as a function of Log relative maturity, $\log_{10}(\tau) = \log_{10}(T\bar{\sigma}^2)$, and moneyness, X_0/K relative to the same domain as in Figure 8.1. We see that the correction is large in the price transition zone and that its maximal value is rather insensitive to the moneyness. We see moreover that when the time to maturity T is large relative to the diffusion time $\bar{\sigma}^{-2}$ then the correction plays a minor role. The red dashed line corresponds to $d_- = 0$, or $\tau = 2\ln(m)$, so that $Q_0^{(1)} = 0$. The blue and red crosses are asymptotic approximations, in $\ln(m)$, for the partial derivative of $Q_0^{(1)}$ with respect to maturity being zero. The blue crosses in the figure are $\tau = 4 + 4\ln(m)$, the red crosses are $\tau = \ln^2(m)$.

d_- and $\theta = t/T$ only:

$$\begin{aligned} \frac{g(d_-)}{K} &= -\frac{d_- \exp(-d_-^2/2)}{\sqrt{2\pi}}, \\ \frac{v(\theta; d_-)}{K^2} &= \frac{1}{2\pi} \int_0^\theta \exp\left(-\frac{d_-^2}{1+s}\right) \frac{ds}{\sqrt{1-s^2}}, \\ \frac{w^H(\theta; d_-)}{K^2} &= \frac{1}{\pi} \int_0^\theta \exp\left(-\frac{d_-^2}{1+s}\right) \frac{(\theta-s)}{(1-s)^2} [2f_4(s, d_-) - f_0(s)] ds - \theta^2 \frac{d_-^2 \exp(-d_-^2)}{2\pi}, \\ w^{\text{BS}}(\theta; d_-) &= -v(\theta; d_-), \\ \frac{w^{\tilde{H}}(\theta; d_-)}{K^2} &= \frac{1}{2\pi} \int_0^\theta \exp\left(-\frac{d_-^2}{1+s}\right) \frac{1}{(1-s)} [f_4(s, d_-) - f_0(s)] ds, \end{aligned}$$

with $f_j, j = 0, 2, 4$ defined in Proposition C.1 and

$$d_- = \frac{\log(X_0/K)}{\sqrt{\tau}} - \frac{\sqrt{\tau}}{2}, \quad \tau = \bar{\sigma}^2 T.$$

It then follows that, as $\varepsilon \rightarrow 0$,

$$\text{Var}(Y_t^{\text{BS}} | \mathcal{F}_0) = \varepsilon \left(\frac{\bar{\Gamma}^2}{\bar{\sigma}^2} - \frac{\rho^2 \bar{D}^2}{\bar{\sigma}^4} \right) v(\theta; d_-) = \left(\frac{\Gamma^2}{\bar{\sigma}^2} - \frac{D^2}{\bar{\sigma}^4} \right) v(\theta; d_-),$$

which gives Eq. (2.9). In Figure 8.5 we plot v as a function of normalized maturity and moneyness. We see that v is large for large exercise times and small values of d_- .

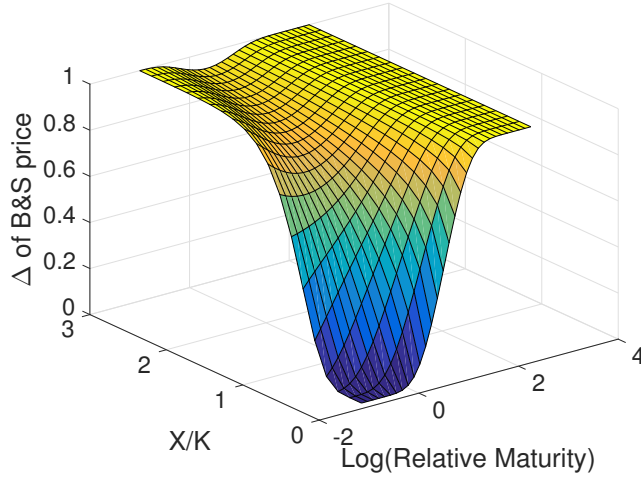


FIG. 8.3. The figure shows the Black-Scholes delta at the effective volatility, that is $\partial_x Q_0^{(0)}$. It is plotted as a function of log relative maturity, $\log_{10}(T\bar{\sigma}^2)$, and moneyness, X_0/K . Relatively far away from the strike this is close to unity corresponding to holding a unit of the underlying in the replicating portfolio. As seen from Figure 8.2 the price correction is small here also.

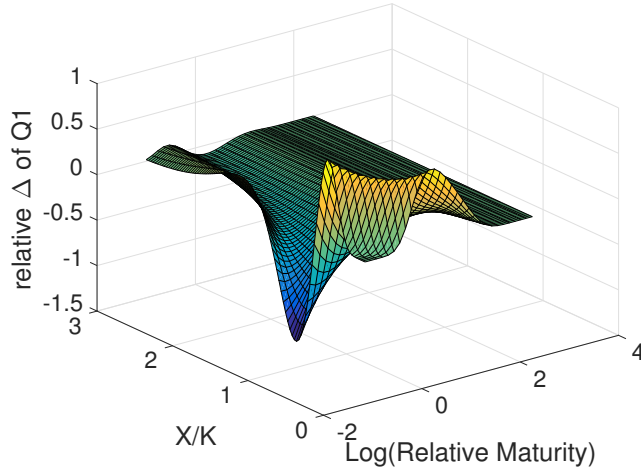


FIG. 8.4. The figure shows the delta of the correction, that is $\bar{\sigma}^2 \partial_x Q_0^{(1)} / \bar{D}$. It is plotted as a function of Log relative maturity, $\log_{10}(T\bar{\sigma}^2)$, and moneyness, X_0/K .

In Figures 8.6 and 8.7 we show respectively w^H and $w^{\tilde{H}}$. In the regime of large exercise times and small values of d_- these schemes offer a slight advantage relative to the (HW) scheme in terms of cost variance. Note that at maturity the two schemes (H) and (\tilde{H}) have the same cost. Recall however that for the scheme (H) it is assumed that the option can be traded at the price $Q^{(0)}$ so the schemes cannot be compared directly other than at maturity when $Q_T^{(1)} = 0$. In Figure 8.8 we show the function g/K which describes the coherent cost correction as a function of d_- , we see that this correction is maximal for d_- around unity.

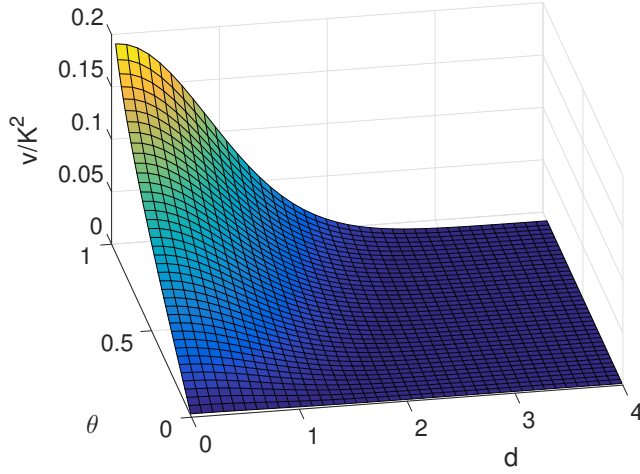


FIG. 8.5. The figure shows the hedging cost variance function $v(\theta; d_-)/K^2$.

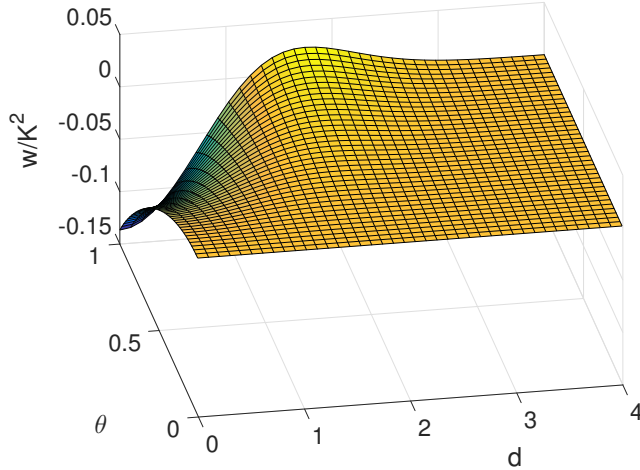


FIG. 8.6. The figure shows the hedging cost variance function $w^H(\theta; d_-)/K^2$.

8.3. Optimality of Practitioners Scheme. The practitioners scheme (BS) has the lowest risk (i.e. cost variance) among the schemes that we have considered (H, HW, BS, \tilde{H}). Here, we show that in fact the practitioners approach is the optimal scheme amongst all DA hedging strategies.

DEFINITION 8.2. A DA hedging scheme is based on a replication portfolio of value P of the form (2.2) with the number of underlyings a_t being a smooth function of t and X_t .

PROPOSITION 8.3. Let $\mathcal{A}(t, x)$ be a smooth and bounded function. Let $a_t = \mathcal{A}(t, X_t)$ be the number of underlyings in a replication portfolio of value $P(t, X_t)$. Let

$$E_t^* = P(t, X_t) - \int_0^t a_s dX_s \quad (8.1)$$

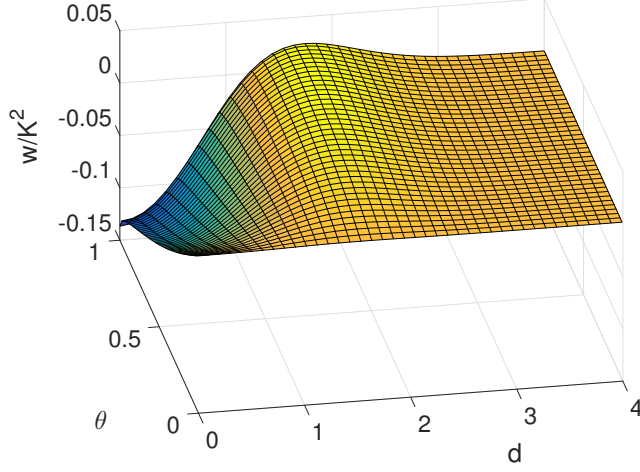


FIG. 8.7. The figure shows the hedging cost variance function $w^{\tilde{H}}(\theta; d_-)/K^2$.

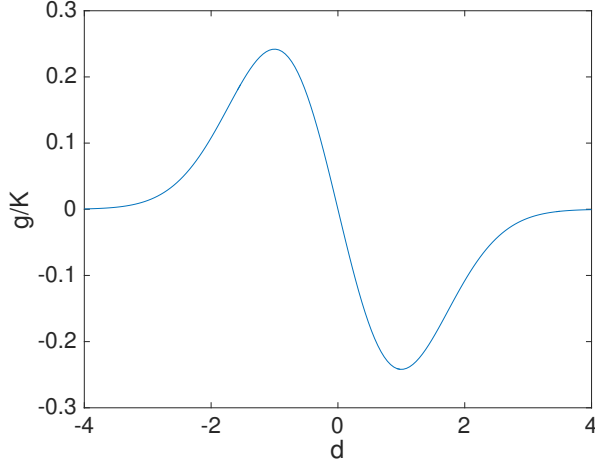


FIG. 8.8. The figure shows the coherent cost correction function $g(d_-)/K$.

be the cost associated to the hedging strategy a_t . Then we have to leading order

$$\mathbb{E}[E_t^* | \mathcal{F}_0] = P(0, X_0), \quad \text{Var}(E_t^* | \mathcal{F}_0) \geq \text{Var}(E_t^{\text{BS}} | \mathcal{F}_0), \quad t \in [0, T]. \quad (8.2)$$

This proposition shows that there is one scheme, the (BS) approach, that is the optimal DA scheme for any exercise time $t \leq T$.

Proof. We write the cost as

$$E_t^* = P(t, X_t) - \int_0^t \delta^{\text{HW}}(s, X_s) dX_s + \int_0^t (\delta^{\text{HW}}(s, X_s) - a_s) dX_s.$$

We first address the most interesting case consistent with the regime addressed here, that is, the case when $\mathcal{A}(t, x) - \partial_x Q_t^{(0)}(x)$ is of order $\sqrt{\varepsilon}$:

$$\mathcal{A}(t, x) = \partial_x Q_t^{(0)}(x) + \sqrt{\varepsilon} \mathcal{A}_1(t, x).$$

Then

$$E_t^* = P(0, X_0) + N_t^{(1)} + \check{N}_t + o(\sqrt{\varepsilon}),$$

with (using Eq. (5.47))

$$\check{N}_t = \sqrt{\varepsilon} \int_0^t \check{A}_s(X_s) \sigma_s^\varepsilon dW_s^*, \quad \check{A}_s(x) = (\rho Q_s^{(1)}(x) - \mathcal{A}_1(s, x))x.$$

The two martingales $N^{(1)}$ and \check{N} have amplitudes of order $\sqrt{\varepsilon}$. Using Eq. (A.14) we get

$$\begin{aligned} \mathbb{E}[(N_t^{(1)})^2 | \mathcal{F}_0] &= \mathbb{E}\left[\int_0^t (x^2 \partial_x^2 Q_s^{(0)})(X_s)^2 (\vartheta_s^\varepsilon)^2 ds | \mathcal{F}_0\right] \\ &= \varepsilon \bar{\Gamma}^2 \mathbb{E}\left[\int_0^t (x^2 \partial_x^2 Q_s^{(0)})(X_s)^2 ds | \mathcal{F}_0\right] + o(\varepsilon), \end{aligned}$$

in the sense that

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E}\left[\left|\varepsilon^{-1} \mathbb{E}[(N_t^{(1)})^2 | \mathcal{F}_0] - \bar{\Gamma}^2 \mathbb{E}\left[\int_0^t (x^2 \partial_x^2 Q_s^{(0)})(X_s)^2 ds | \mathcal{F}_0\right]\right|\right] = 0.$$

Similarly, using Eqs. (A.13) and (A.19),

$$\begin{aligned} \mathbb{E}[(\check{N}_t)^2 | \mathcal{F}_0] &= \varepsilon \mathbb{E}\left[\int_0^t \check{A}_s(X_s)^2 (\sigma_s^\varepsilon)^2 ds | \mathcal{F}_0\right] \\ &= \varepsilon \bar{\sigma}^2 \mathbb{E}\left[\int_0^t \check{A}_s(X_s)^2 ds | \mathcal{F}_0\right] + o(\varepsilon), \\ \mathbb{E}[N_t^{(1)} \check{N}_t | \mathcal{F}_0] &= \sqrt{\varepsilon} \rho \mathbb{E}\left[\int_0^t ((x^2 \partial_x^2 Q_s^{(0)}) \check{A}_s)(X_s) \sigma_s^\varepsilon \vartheta_s^\varepsilon ds | \mathcal{F}_0\right] \\ &= \varepsilon \rho \bar{D} \mathbb{E}\left[\int_0^t ((x^2 \partial_x^2 Q_s^{(0)}) \check{A}_s)(X_s) ds | \mathcal{F}_0\right] + o(\varepsilon). \end{aligned}$$

Therefore, we find to leading order

$$\check{\rho}_t = \text{Corr}(N_t^{(1)}, \check{N}_t | \mathcal{F}_0) = \frac{\rho \bar{D} \mathbb{E}\left[\int_0^t ((x^2 \partial_x^2 Q_s^{(0)}) \check{A}_s)(X_s) ds | \mathcal{F}_0\right]}{\bar{\sigma} \bar{\Gamma} \sqrt{\mathbb{E}\left[\int_0^t (x^2 \partial_x^2 Q_s^{(0)})(X_s)^2 ds | \mathcal{F}_0\right] \mathbb{E}\left[\int_0^t \check{A}_s(X_s)^2 ds | \mathcal{F}_0\right]}},$$

so that by Cauchy-Schwarz inequality $|\check{\rho}_t| \leq \bar{\rho}$, where

$$\bar{\rho} = \frac{\rho \bar{D}}{\bar{\sigma} \bar{\Gamma}} = \frac{D}{\bar{\sigma} \bar{\Gamma}}. \quad (8.3)$$

Thus, using Proposition 8.1 and denoting

$$\alpha_t = \sqrt{\frac{\text{Var}(\check{N}_t | \mathcal{F}_0)}{\text{Var}(N_t^{(1)} | \mathcal{F}_0)}},$$

we have

$$\begin{aligned} \text{Var}(E_t^* | \mathcal{F}_0) &= \text{Var}(N_t^{(1)} | \mathcal{F}_0) (1 + 2\check{\rho}_t \alpha_t + \alpha_t^2) \geq \text{Var}(N_t^{(1)} | \mathcal{F}_0) (1 - 2\bar{\rho} \alpha_t + \alpha_t^2) \\ &\geq \text{Var}(N_t^{(1)} | \mathcal{F}_0) (1 - \bar{\rho}^2) = \text{Var}(E_t^{\text{BS}} | \mathcal{F}_0), \end{aligned}$$

which proves the desired result.

If we assume that $\mathcal{A}(t, x) - \partial_x Q_t^{(0)}(x)$ is smaller than $\sqrt{\varepsilon}$, then we easily find that $E_t^* = P(0, X_0) + N_t^{(1)} + o(\sqrt{\varepsilon})$, and therefore $\text{Var}(E_t^* | \mathcal{F}_0) = \text{Var}(N_t^{(1)} | \mathcal{F}_0)$ up to terms of order $o(\varepsilon)$.

If we assume that $\mathcal{A}(t, x) - \partial_x Q_t^{(0)}(x)$ is larger than $\sqrt{\varepsilon}$:

$$\mathcal{A}(t, x) = \partial_x Q_t^{(0)}(x) + \varepsilon^p \mathcal{A}_1(t, x),$$

with $p \in [0, 1/2)$, then

$$E_t^* = P(0, X_0) + \hat{N}_t + o(\varepsilon^p),$$

with

$$\hat{N}_t = \varepsilon^p \int_0^t \hat{\mathcal{A}}_s(X_s) \sigma_s^\varepsilon dW_s^*, \quad \hat{\mathcal{A}}_s(x) = -\mathcal{A}_1(s, x)x.$$

We then have

$$\mathbb{E}[(N_t^{(1)})^2 | \mathcal{F}_0] = O(\varepsilon), \quad \mathbb{E}[(\hat{N}_t)^2 | \mathcal{F}_0] = \bar{\sigma}^2 \varepsilon^{2p} \mathbb{E}\left[\int_0^t \hat{\mathcal{A}}_s(X_s)^2 ds | \mathcal{F}_0\right] (1 + o(1)),$$

which shows that

$$\text{Var}(E_t^* | \mathcal{F}_0) = \text{Var}(\hat{N}_t | \mathcal{F}_0) (1 + o(1)) \gg \text{Var}(N_t^{(1)} | \mathcal{F}_0) \geq \text{Var}(E_t^{\text{BS}} | \mathcal{F}_0).$$

For completeness (and to prove the last inequality in (2.9)), we also remark that, by using Eq. (6.1) and Lemma A.6, we have

$$|\bar{\rho}| = \frac{|\rho|}{\bar{\sigma}} \lim_{\varepsilon \rightarrow 0} \frac{\mathbb{E}[\vartheta_t^\varepsilon \sigma_t^\varepsilon]}{\sqrt{\mathbb{E}[(\vartheta_t^\varepsilon)^2]}} \leq |\rho| \frac{1}{\bar{\sigma}} \lim_{\varepsilon \rightarrow 0} \sqrt{\mathbb{E}[(\sigma_t^\varepsilon)^2]} = |\rho|.$$

□

9. Conclusions. Classic price replicating delta hedging strategies are important in hedging practice. We present here a novel analysis of the extra hedging cost associated with such schemes that follows from a stochastic volatility situation and thus an incomplete market context. We model the volatility as a stationary stochastic process that is rapidly mean-reverting relative to the diffusion time of the underlying. Specifically, the volatility is a smooth function of a Volterra type Gaussian process (an integral of a standard Brownian motion with respect to a deterministic integral kernel). We incorporate leverage in our modeling so that the Brownian motion driving the volatility is correlated with the Brownian motion driving the underlying.

In this context we identify the correction to the price that is produced by the stochastic volatility. The two market parameters that determine this correction are the *effective volatility* or root mean square volatility and a *market pricing parameter*. The hedging cost incurred due to the stochastic nature of the volatility is characterized by a vega risk martingale. The amplitude of this martingale is proportional to a *market risk parameter* that needs to be calibrated to the market in order to quantify the hedging cost (mean and variance). It is interesting to observe that this market risk parameter cannot be identified from the implied volatility skew.

We consider specifically hedging of a European option and then we get explicit expressions for the hedging cost. We consider a large class of dynamic asset based

hedging schemes (which contains all delta hedging strategies) and we find that the optimal scheme is the delta hedging scheme where the delta is the Black-Scholes delta when evaluated at the implied volatility, the so-called “practitioners delta”. All the hedging schemes that we consider can be implemented without knowledge of the market risk parameter, only the quantitative evaluation of the hedging cost requires the knowledge of the market risk parameter. In the case of no leverage, the market pricing parameter referred to above is zero, all schemes coincide, and the hedging cost is determined by the vega risk martingale. For general leverage and for each choice of delta we identify the *hedging risk surface* which characterizes the variance of the cost.

Note that we have assumed a smooth bounded payoff in the proofs of our results, although the formulas can be applied with a more general payoff. The proofs for nonsmooth payoff functions are more involved than the corresponding ones dedicated to pricing as presented in [11], they should involve a payoff regularization scheme and they will be presented elsewhere.

Finally, let us remark that we have considered a simplified market situation. In order to capture a more general market context other effects, like transaction cost, discreteness, market price of volatility risk and non-zero interest rate need to be taken into account. Here, we wanted to characterize in a rigorous way the effect of market incompleteness in the simple albeit practically important context of delta hedging schemes leaving for future work more sophisticated hedging schemes incorporating in particular other derivatives.

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Appendix A. Effective Market Lemmas.

We denote

$$G(z) = \frac{1}{2}(F(z)^2 - \bar{\sigma}^2). \quad (\text{A.1})$$

The random term ϕ_t^ε defined by (5.18) has the form

$$\phi_t^\varepsilon = \mathbb{E} \left[\int_t^T G(Z_s^\varepsilon) ds \mid \mathcal{F}_t \right]. \quad (\text{A.2})$$

The martingale ψ_t^ε defined by (5.15) has the form

$$\psi_t^\varepsilon = \mathbb{E} \left[\int_0^T G(Z_s^\varepsilon) ds \mid \mathcal{F}_t \right]. \quad (\text{A.3})$$

LEMMA A.1. *For any smooth function f with bounded derivative, we have*

$$\text{Var}(\mathbb{E}[f(Z_t^\varepsilon) \mid \mathcal{F}_0]) \leq \|f'\|_\infty^2 (\sigma_{t,\infty}^\varepsilon)^2, \quad (\text{A.4})$$

where we have defined for any $0 \leq t \leq s \leq \infty$:

$$(\sigma_{t,s}^\varepsilon)^2 = \sigma_z^2 \int_t^s \mathcal{K}^\varepsilon(u)^2 du. \quad (\text{A.5})$$

Proof. The conditional distribution of Z_t^ε given \mathcal{F}_0 is Gaussian with mean

$$\mathbb{E}[Z_t^\varepsilon \mid \mathcal{F}_0] = \sigma_z \int_{-\infty}^0 \mathcal{K}^\varepsilon(t-u) dW_u \quad (\text{A.6})$$

and variance

$$\text{Var}(Z_t^\varepsilon | \mathcal{F}_0) = (\sigma_{0,t}^\varepsilon)^2 = \sigma_z^2 \int_0^t \mathcal{K}^\varepsilon(u)^2 du. \quad (\text{A.7})$$

Therefore

$$\text{Var}(\mathbb{E}[f(Z_t^\varepsilon) | \mathcal{F}_0]) = \text{Var}\left(\int_{\mathbb{R}} f(\mathbb{E}[Z_t^\varepsilon | \mathcal{F}_0] + \sigma_{0,t}^\varepsilon z) p(z) dz\right),$$

where $p(z)$ is the pdf of the standard normal distribution. By (A.6) the random variable $\mathbb{E}[Z_t^\varepsilon | \mathcal{F}_0]$ is Gaussian with mean zero and variance $(\sigma_{t,\infty}^\varepsilon)^2$ so that

$$\begin{aligned} \text{Var}(\mathbb{E}[f(Z_t^\varepsilon) | \mathcal{F}_0]) &= \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} dz dz' p(z) p(z') \int_{\mathbb{R}} \int_{\mathbb{R}} du du' p(u) p(u') \\ &\quad \times \left[f(\sigma_{t,\infty}^\varepsilon u + \sigma_{0,t}^\varepsilon z) - f(\sigma_{t,\infty}^\varepsilon u' + \sigma_{0,t}^\varepsilon z) \right] \\ &\quad \times \left[f(\sigma_{t,\infty}^\varepsilon u + \sigma_{0,t}^\varepsilon z') - f(\sigma_{t,\infty}^\varepsilon u' + \sigma_{0,t}^\varepsilon z') \right] \\ &\leq \|f'\|_\infty^2 (\sigma_{t,\infty}^\varepsilon)^2 \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} du du' p(u) p(u') (u - u')^2 \\ &= \|f'\|_\infty^2 (\sigma_{t,\infty}^\varepsilon)^2, \end{aligned}$$

which is the desired result. \square

LEMMA A.2. For any $t \leq T$, ϕ_t^ε is a zero-mean random variable with standard deviation of order $\varepsilon^{(d-\frac{1}{2}) \wedge 1}$:

$$\sup_{\varepsilon \in (0,1]} \sup_{t \in [0,T]} \varepsilon^{(2d-1) \wedge 2} \mathbb{E}[(\phi_t^\varepsilon)^2] < \infty. \quad (\text{A.8})$$

Proof. For $t \in [0, T]$ the second moment of ϕ_t^ε is:

$$\begin{aligned} \mathbb{E}[(\phi_t^\varepsilon)^2] &= \mathbb{E}\left[\mathbb{E}\left[\int_t^T G(Z_s^\varepsilon) ds \mid \mathcal{F}_t\right]^2\right] \\ &= \int_0^{T-t} ds \int_0^{T-t} ds' \text{Cov}(\mathbb{E}[G(Z_s^\varepsilon) | \mathcal{F}_0], \mathbb{E}[G(Z_{s'}^\varepsilon) | \mathcal{F}_0]). \end{aligned}$$

We have by Lemma A.1

$$\mathbb{E}[(\phi_t^\varepsilon)^2] \leq \left(\int_0^{T-t} ds (\text{Var}(\mathbb{E}[G(Z_s^\varepsilon) | \mathcal{F}_0]))^{1/2}\right)^2 \leq \|G'\|_\infty^2 \left(\int_0^{T-t} ds \sigma_{s,\infty}^\varepsilon\right)^2.$$

In view of Lemma A.10 we then have

$$\mathbb{E}[(\phi_t^\varepsilon)^2] \leq C_T (\varepsilon + \varepsilon^{d-\frac{1}{2}})^2 \leq 4C_T \varepsilon^{(2d-1) \wedge 2},$$

uniformly in $t \leq T$ and $\varepsilon \in (0, 1]$ for some constant C_T . \square

LEMMA A.3. Let Y_t be a bounded adapted process, we have

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1/2} \sup_{t \in [0,T]} \mathbb{E}\left[\left|\int_0^t Y_s \phi_s^\varepsilon dW_s^*\right|^2 \mid \mathcal{F}_0\right]^{1/2} = 0. \quad (\text{A.9})$$

Proof. We have by the Itô isometry

$$\mathbb{E} \left[\left| \int_0^t Y_s \phi_s^\varepsilon dW_s^* \right|^2 \mid \mathcal{F}_0 \right] = \mathbb{E} \left[\int_0^t |Y_s \phi_s^\varepsilon|^2 ds \mid \mathcal{F}_0 \right],$$

and the result then follows from Lemma A.2 noting that we consider the case $d > 1$.
□

We next present a result regarding the quadratic variation of ψ^ε .

LEMMA A.4. $(\psi_t^\varepsilon)_{t \in [0, T]}$ is a square-integrable martingale and

$$d \langle \psi^\varepsilon, W \rangle_t = \vartheta_t^\varepsilon dt, \quad d \langle \psi^\varepsilon, \psi^\varepsilon \rangle_t = (\vartheta_t^\varepsilon)^2 dt, \quad (\text{A.10})$$

with

$$\vartheta_t^\varepsilon = \sigma_z \int_t^T \mathbb{E} [G'(Z_s^\varepsilon) \mid \mathcal{F}_t] \mathcal{K}^\varepsilon(s-t) ds. \quad (\text{A.11})$$

An alternative expression of ϑ_t^ε is given in (A.12).

Proof. This follows from [12, Lemma B.1] and its proof. For $t < s$, the conditional distribution of Z_s^ε given \mathcal{F}_t is Gaussian with mean

$$\mathbb{E}[Z_s^\varepsilon \mid \mathcal{F}_t] = \sigma_z \int_{-\infty}^t \mathcal{K}^\varepsilon(s-u) dW_u$$

and deterministic variance given by

$$\text{Var}(Z_s^\varepsilon \mid \mathcal{F}_t) = (\sigma_{0, s-t}^\varepsilon)^2,$$

where $\sigma_{s,t}^\varepsilon$ is defined by (A.5). Therefore we have

$$\mathbb{E}[G(Z_s^\varepsilon) \mid \mathcal{F}_t] = \int_{\mathbb{R}} G\left(\sigma_z \int_{-\infty}^t \mathcal{K}^\varepsilon(s-u) dW_u + \sigma_{0, s-t}^\varepsilon z\right) p(z) dz,$$

where $p(z)$ is the pdf of the standard normal distribution. As a random process in t it is a continuous martingale. By Itô's formula, for any $t < s$:

$$\begin{aligned} \mathbb{E}[G(Z_s^\varepsilon) \mid \mathcal{F}_t] &= \int_{\mathbb{R}} G\left(\sigma_z \int_{-\infty}^0 \mathcal{K}^\varepsilon(s-v) dW_v + \sigma_{0, s}^\varepsilon z\right) p(z) dz \\ &+ \int_0^t \int_{\mathbb{R}} G'\left(\sigma_z \int_{-\infty}^u \mathcal{K}^\varepsilon(s-v) dW_v + \sigma_{0, s-u}^\varepsilon z\right) z p(z) dz \partial_u \sigma_{0, s-u}^\varepsilon du \\ &+ \sigma_z \int_0^t \int_{\mathbb{R}} G'\left(\sigma_z \int_{-\infty}^u \mathcal{K}^\varepsilon(s-v) dW_v + \sigma_{0, s-u}^\varepsilon z\right) p(z) dz \mathcal{K}^\varepsilon(s-u) dW_u \\ &+ \frac{\sigma_z^2}{2} \int_0^t \int_{\mathbb{R}} G''\left(\sigma_z \int_{-\infty}^u \mathcal{K}^\varepsilon(s-v) dW_v + \sigma_{0, s-u}^\varepsilon z\right) p(z) dz \mathcal{K}^\varepsilon(s-u)^2 du. \end{aligned}$$

Note that we have from Eq. (A.5) that

$$2\sigma_{0, s-u}^\varepsilon \partial_u \sigma_{0, s-u}^\varepsilon = -\partial_s (\sigma_{0, s-u}^\varepsilon)^2 = -\sigma_z^2 \mathcal{K}^\varepsilon(s-u)^2.$$

The martingale representation then follows explicitly via integration by parts (with respect to z , using $zp(z) = -\partial_z p(z)$):

$$\begin{aligned}\mathbb{E}[G(Z_s^\varepsilon) \mid \mathcal{F}_t] &= \int_{\mathbb{R}} G\left(\sigma_z \int_{-\infty}^0 \mathcal{K}^\varepsilon(s-v) dW_v + \sigma_{0,s}^\varepsilon\right) p(z) dz \\ &\quad + \sigma_z \int_0^t \int_{\mathbb{R}} G'\left(\sigma_z \int_{-\infty}^u \mathcal{K}^\varepsilon(s-v) dW_v + \sigma_{0,s-u}^\varepsilon\right) p(z) dz \mathcal{K}^\varepsilon(s-u) dW_u.\end{aligned}$$

We also have

$$\begin{aligned}G(Z_s^\varepsilon) &= G\left(\sigma_z \int_{-\infty}^s \mathcal{K}^\varepsilon(s-v) dW_v\right) \\ &= \int_{\mathbb{R}} G\left(\sigma_z \int_{-\infty}^u \mathcal{K}^\varepsilon(s-v) dW_v + \sigma_{0,s-u}^\varepsilon\right) p(z) dz \Big|_{u=s} \\ &= \int_{\mathbb{R}} G\left(\sigma_z \int_{-\infty}^0 \mathcal{K}^\varepsilon(s-v) dW_v + \sigma_{0,s}^\varepsilon\right) p(z) dz \\ &\quad + \int_0^s \int_{\mathbb{R}} G'\left(\sigma_z \int_{-\infty}^u \mathcal{K}^\varepsilon(s-v) dW_v + \sigma_{0,s-u}^\varepsilon\right) zp(z) dz \partial_u \sigma_{0,s-u}^\varepsilon du \\ &\quad + \sigma_z \int_0^s \int_{\mathbb{R}} G'\left(\sigma_z \int_{-\infty}^u \mathcal{K}^\varepsilon(s-v) dW_v + \sigma_{0,s-u}^\varepsilon\right) p(z) dz \mathcal{K}^\varepsilon(s-u) dW_u \\ &\quad + \frac{\sigma_z^2}{2} \int_0^s \int_{\mathbb{R}} G''\left(\sigma_z \int_{-\infty}^u \mathcal{K}^\varepsilon(s-v) dW_v + \sigma_{0,s-u}^\varepsilon\right) p(z) dz \mathcal{K}^\varepsilon(s-u)^2 du \\ &= \int_{\mathbb{R}} G\left(\sigma_z \int_{-\infty}^0 \mathcal{K}^\varepsilon(s-v) dW_v + \sigma_{0,s}^\varepsilon\right) p(z) dz \\ &\quad + \sigma_z \int_0^s \int_{\mathbb{R}} G'\left(\sigma_z \int_{-\infty}^u \mathcal{K}^\varepsilon(s-v) dW_v + \sigma_{0,s-u}^\varepsilon\right) p(z) dz \mathcal{K}^\varepsilon(s-u) dW_u.\end{aligned}$$

Therefore

$$\begin{aligned}\psi_t^\varepsilon &= \int_0^t G(Z_s^\varepsilon) ds + \int_t^T \mathbb{E}[G(Z_s^\varepsilon) \mid \mathcal{F}_t] ds \\ &= \left[\int_{\mathbb{R}} \int_0^T G\left(\sigma_z \int_{-\infty}^0 \mathcal{K}^\varepsilon(s-v) dW_v + \sigma_{0,s}^\varepsilon\right) ds p(z) dz \right] \\ &\quad + \sigma_z \int_0^t \left[\int_u^T \int_{\mathbb{R}} G'\left(\sigma_z \int_{-\infty}^u \mathcal{K}^\varepsilon(s-v) dW_v + \sigma_{0,s-u}^\varepsilon\right) p(z) dz \mathcal{K}^\varepsilon(s-u) ds \right] dW_u.\end{aligned}$$

This gives (A.10) with

$$\vartheta_t^\varepsilon = \sigma_z \int_t^T \int_{\mathbb{R}} G'\left(\sigma_z \int_{-\infty}^t \mathcal{K}^\varepsilon(s-v) dW_v + \sigma_{0,s-t}^\varepsilon\right) p(z) dz \mathcal{K}^\varepsilon(s-t) ds, \quad (\text{A.12})$$

which can also be written as stated in the Lemma. \square

LEMMA A.5. *Let Y_t be a bounded adapted process. Then we have*

$$\sup_{\varepsilon \in (0,1]} \varepsilon^{-1/2} \sup_{t \in [0,T]} \mathbb{E} \left[\left| \int_0^t Y_s d\psi_s^\varepsilon \right|^2 \mid \mathcal{F}_0 \right]^{1/2} < \infty.$$

Proof. There exists $\tilde{K} < \infty$ such that, for $t \in (0, T)$,

$$\mathbb{E} \left[\left| \int_0^t Y_s d\psi_s^\varepsilon \right|^2 \mid \mathcal{F}_0 \right] \leq \tilde{K} \mathbb{E} [\langle \psi^\varepsilon, \psi^\varepsilon \rangle_T - \langle \psi^\varepsilon, \psi^\varepsilon \rangle_0 \mid \mathcal{F}_0],$$

and the result follows from (A.10) and (5.20). \square

LEMMA A.6. *Let $f(t, x)$ be smooth bounded and with bounded derivatives and let X_t be defined by Eq. (3.1). Then for any $t \in [0, T]$ we have*

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\left(\int_0^t f(s, X_s) (\varepsilon^{-1/2} \sigma_s^\varepsilon \vartheta_s^\varepsilon - \bar{D}) ds \right)^2 \right] = 0, \quad (\text{A.13})$$

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\left(\int_0^t f(s, X_s) (\varepsilon^{-1} (\vartheta_s^\varepsilon)^2 - \bar{\Gamma}^2) ds \right)^2 \right] = 0. \quad (\text{A.14})$$

Proof. The result in Eq. (A.13) follows via an argument as in the proof of Eq. (5.28) for $j = 3$ as given in [13] (note that, by (5.20), $\varepsilon^{-1/2} \vartheta_t^\varepsilon$ is uniformly bounded almost surely). The result in Eq. (A.14) follows via an argument as in the proof of Eq. (5.28) for $j = 2$ as given in [13]. To complete that proof it remains to show that

$$\lim_{\varepsilon \rightarrow 0} \sup_{t \in [0, T]} \mathbb{E} \left[(\kappa_t^\varepsilon)^2 \right] = 0, \quad \text{for } \kappa_t^\varepsilon = \int_0^t (\varepsilon^{-1} (\vartheta_s^\varepsilon)^2 - \bar{\Gamma}) ds.$$

We show this in Lemma A.7. \square

LEMMA A.7. *Let*

$$\kappa_t^\varepsilon = \int_0^t (\varepsilon^{-1} (\vartheta_s^\varepsilon)^2 - \bar{\Gamma}) ds,$$

then

$$\lim_{\varepsilon \rightarrow 0} \sup_{t \in [0, T]} \mathbb{E} \left[(\kappa_t^\varepsilon)^2 \right] = 0.$$

Proof. As $(a + b)^2 \leq 2a^2 + 2b^2$ we have

$$\mathbb{E} \left[(\kappa_t^\varepsilon)^2 \right] \leq 2\varepsilon^{-2} \int_0^t ds \int_0^t ds' \text{Cov} \left((\vartheta_s^\varepsilon)^2, (\vartheta_{s'}^\varepsilon)^2 \right) + 2 \left(\int_0^t (\varepsilon^{-1} \mathbb{E} [(\vartheta_s^\varepsilon)^2] - \bar{\Gamma}) ds \right)^2.$$

The results then follows from Lemmas A.8 and A.9 and the bound in Eq. (5.20) using dominated convergence theorem. \square

LEMMA A.8. *Let ϑ_t^ε be defined by (A.11). We have for any $t \in [0, T]$:*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} \mathbb{E} \left[(\vartheta_t^\varepsilon)^2 \right] = \bar{\Gamma}^2,$$

where $\bar{\Gamma}$ is defined by (5.41).

Proof. We consider

$$\begin{aligned} \mathbb{E} \left[(\vartheta_t^\varepsilon)^2 \right] \sigma_z^{-2} &= \mathbb{E} \left[\mathbb{E} \left[\int_t^T G'(Z_s^\varepsilon) \mathcal{K}^\varepsilon(s-t) ds \mid \mathcal{F}_t \right] \mathbb{E} \left[\int_t^T G'(Z_s^\varepsilon) \mathcal{K}^\varepsilon(s-t) ds \mid \mathcal{F}_t \right] \right] \\ &= 2 \int_0^{T-t} ds \int_s^{T-t} ds' \mathbb{E} \left[\mathbb{E} [G'(Z_s^\varepsilon) \mid \mathcal{F}_0] \mathbb{E} [G'(Z_{s'}^\varepsilon) \mid \mathcal{F}_0] \right] \mathcal{K}^\varepsilon(s) \mathcal{K}^\varepsilon(s'). \end{aligned}$$

We can then write

$$\begin{aligned}
\mathbb{E} \left[(\vartheta_t^\varepsilon)^2 \right] \sigma_z^{-2} &= 2 \int_0^{T-t} ds \int_s^{T-t} ds' \\
&\quad \times \mathbb{E} \left[\left[\int_{\mathbb{R}} G'(\sigma_z \int_{-\infty}^0 \mathcal{K}^\varepsilon(s-v) dW_v + \sigma_{0,s}^\varepsilon z) p(z) dz \right] \right. \\
&\quad \times \left. \left[\int_{\mathbb{R}} G'(\sigma_z \int_{-\infty}^0 \mathcal{K}^\varepsilon(s'-v) dW_v + \sigma_{0,s'}^\varepsilon z') p(z') dz' \right] \right] \mathcal{K}^\varepsilon(s) \mathcal{K}^\varepsilon(s') \\
&= 2 \int_0^{T-t} ds \int_s^{T-t} ds' \int_{\mathbb{R}^2} dud u' \left[\int_{\mathbb{R}} G'(\sigma_{s,\infty}^\varepsilon u + \sigma_{0,s}^\varepsilon z) p(z) dz \right] \\
&\quad \times \left[\int_{\mathbb{R}} G'(\sigma_{s',\infty}^\varepsilon u' + \sigma_{0,s'}^\varepsilon z') p(z') dz' \right] p_{\tilde{\mathcal{C}}_{\mathcal{K}}^\varepsilon(s,s')}(u, u') \mathcal{K}^\varepsilon(s) \mathcal{K}^\varepsilon(s'),
\end{aligned}$$

where $p(z)$ is the pdf of the standard normal distribution, p_C is the pdf of the (standardized) bivariate normal distribution with mean zero and covariance matrix as in Lemma 4.1, and

$$\tilde{\mathcal{C}}_{\mathcal{K}}^\varepsilon(s, s') = \frac{\sigma_z^2 \int_{-\infty}^0 \mathcal{K}^\varepsilon(s'-v) \mathcal{K}^\varepsilon(s-v) dv}{\sigma_{s,\infty}^\varepsilon \sigma_{s',\infty}^\varepsilon}.$$

By remarking that $(\sigma_{s,\infty}^\varepsilon)^2 + (\sigma_{0,s}^\varepsilon)^2 = \sigma_z^2$ and $\sigma_{s,\infty}^\varepsilon \sigma_{s',\infty}^\varepsilon \tilde{\mathcal{C}}_{\mathcal{K}}^\varepsilon(s, s') = \sigma_z^2 \mathcal{C}_{\mathcal{K}}^\varepsilon(s, s')$, with $\mathcal{C}_{\mathcal{K}}^\varepsilon(s, s') = \mathcal{C}_{\mathcal{K}}(s/\varepsilon, s'/\varepsilon)$, we can see that, if (Z, Z', U, U') is a four-dimensional Gaussian vector with pdf $p(z)p(z')p_{\tilde{\mathcal{C}}_{\mathcal{K}}^\varepsilon(s,s')}(u, u')$, then $(\sigma_{s,\infty}^\varepsilon U + \sigma_{0,s}^\varepsilon Z, \sigma_{s',\infty}^\varepsilon U' + \sigma_{0,s'}^\varepsilon Z') = (\sigma_z Y, \sigma_z Y')$ where (Y, Y') is a two-dimensional Gaussian vector with pdf $p_{\mathcal{C}_{\mathcal{K}}^\varepsilon(s,s')}(y, y')$. This gives

$$\mathbb{E} \left[(\vartheta_t^\varepsilon)^2 \right] = 2\sigma_z^2 \int_0^{T-t} ds \int_s^{T-t} ds' \int_{\mathbb{R}^2} dy dy' G'(\sigma_z y) G'(\sigma_z y') p_{\mathcal{C}_{\mathcal{K}}^\varepsilon(s,s')}(y, y') \mathcal{K}^\varepsilon(s) \mathcal{K}^\varepsilon(s'),$$

or

$$\mathbb{E} \left[(\vartheta_t^\varepsilon)^2 \right] = 2\varepsilon \sigma_z^2 \int_0^{\frac{T-t}{\varepsilon}} ds \int_s^{\frac{T-t}{\varepsilon}} ds' \int_{\mathbb{R}^2} dy dy' G'(\sigma_z y) G'(\sigma_z y') p_{\mathcal{C}_{\mathcal{K}}(s,s')}(y, y') \mathcal{K}(s) \mathcal{K}(s').$$

By using the fact that $\mathcal{K} \in L^1(0, \infty)$ we finally get

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} \mathbb{E} \left[(\vartheta_t^\varepsilon)^2 \right] = \bar{\Gamma}^2,$$

with the expression (5.41) of $\bar{\Gamma}$, which completes the proof of the Lemma. \square

LEMMA A.9. *For any $0 \leq t < t' < T$ we have*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-2} \left| \text{Cov} \left((\vartheta_t^\varepsilon)^2, (\vartheta_{t'}^\varepsilon)^2 \right) \right| = 0.$$

Proof. Let us consider $0 \leq t' < t \leq T$. We have

$$\begin{aligned}
\mathbb{E} \left[(\vartheta_t^\varepsilon)^2 (\vartheta_{t'}^\varepsilon)^2 \right] &= \sigma_z^4 \int_t^T ds \mathcal{K}^\varepsilon(s-t) \int_t^T ds' \mathcal{K}^\varepsilon(s'-t) \int_{t'}^T du \mathcal{K}^\varepsilon(u-t') \\
&\quad \times \int_{t'}^T du' \mathcal{K}^\varepsilon(u'-t') \mathbb{E} \left[\mathbb{E} [G'(Z_s^\varepsilon) | \mathcal{F}_t] \mathbb{E} [G'(Z_{s'}^\varepsilon) | \mathcal{F}_t] \mathbb{E} [G'(Z_u^\varepsilon) | \mathcal{F}_{t'}] \mathbb{E} [G'(Z_{u'}^\varepsilon) | \mathcal{F}_{t'}] \right],
\end{aligned}$$

so we can write

$$\begin{aligned}
\text{Cov}((\vartheta_t^\varepsilon)^2, (\vartheta_{t'}^\varepsilon)^2) &= \sigma_z^4 \int_t^T ds \mathcal{K}^\varepsilon(s-t) \int_t^T ds' \mathcal{K}^\varepsilon(s'-t) \\
&\times \int_{t'}^T du \mathcal{K}^\varepsilon(u-t') \int_{t'}^T du' \mathcal{K}^\varepsilon(u'-t') \left\{ \mathbb{E} \left[\mathbb{E}[G'(Z_s^\varepsilon) | \mathcal{F}_t] \mathbb{E}[G'(Z_{s'}^\varepsilon) | \mathcal{F}_t] \right. \right. \\
&\times \mathbb{E}[G'(Z_u^\varepsilon) | \mathcal{F}_{t'}] \mathbb{E}[G'(Z_{u'}^\varepsilon) | \mathcal{F}_{t'}] \left. \right] - \mathbb{E} \left[\mathbb{E}[G'(Z_s^\varepsilon) | \mathcal{F}_t] \mathbb{E}[G'(Z_{s'}^\varepsilon) | \mathcal{F}_t] \right] \\
&\times \mathbb{E} \left[\mathbb{E}[G'(Z_u^\varepsilon) | \mathcal{F}_{t'}] \mathbb{E}[G'(Z_{u'}^\varepsilon) | \mathcal{F}_{t'}] \right] \left. \right\} \\
&= \sigma_z^4 \int_t^T ds \mathcal{K}^\varepsilon(s-t) \int_t^T ds' \mathcal{K}^\varepsilon(s'-t) \\
&\times \int_{t'}^T du \mathcal{K}^\varepsilon(u-t') \int_{t'}^T du' \mathcal{K}^\varepsilon(u'-t') \left\{ \mathbb{E} \left[\mathbb{E} \left[\mathbb{E}[G'(Z_s^\varepsilon) | \mathcal{F}_t] \mathbb{E}[G'(Z_{s'}^\varepsilon) | \mathcal{F}_t] \middle| \mathcal{F}_{t'} \right] \right. \right. \\
&\times \mathbb{E}[G'(Z_u^\varepsilon) | \mathcal{F}_{t'}] \mathbb{E}[G'(Z_{u'}^\varepsilon) | \mathcal{F}_{t'}] \left. \right] - \mathbb{E} \left[\mathbb{E} \left[\mathbb{E}[G'(Z_s^\varepsilon) | \mathcal{F}_t] \mathbb{E}[G'(Z_{s'}^\varepsilon) | \mathcal{F}_t] \right] \right. \\
&\times \mathbb{E}[G'(Z_u^\varepsilon) | \mathcal{F}_{t'}] \mathbb{E}[G'(Z_{u'}^\varepsilon) | \mathcal{F}_{t'}] \left. \right] \left. \right\},
\end{aligned}$$

and therefore

$$\begin{aligned}
|\text{Cov}((\vartheta_t^\varepsilon)^2, (\vartheta_{t'}^\varepsilon)^2)| &\leq \sigma_z^4 \|G'\|_\infty^2 \int_t^T ds |\mathcal{K}^\varepsilon(s-t)| \int_t^T ds' |\mathcal{K}^\varepsilon(s'-t)| \\
&\times \int_{t'}^T du |\mathcal{K}^\varepsilon(u-t')| \int_{t'}^T du' |\mathcal{K}^\varepsilon(u'-t')| \mathbb{E} \left[\left(\mathbb{E} \left[\mathbb{E}[G'(Z_s^\varepsilon) | \mathcal{F}_t] \mathbb{E}[G'(Z_{s'}^\varepsilon) | \mathcal{F}_t] \middle| \mathcal{F}_{t'} \right] \right. \right. \\
&\left. \left. - \mathbb{E} \left[\mathbb{E}[G'(Z_s^\varepsilon) | \mathcal{F}_t] \mathbb{E}[G'(Z_{s'}^\varepsilon) | \mathcal{F}_t] \right] \right)^2 \right]^{1/2}.
\end{aligned}$$

We can write for any $\tau > t > t'$:

$$\begin{aligned}
Z_\tau^\varepsilon &= A_{t',\tau}^\varepsilon + B_{t',t,\tau}^\varepsilon + C_{t,\tau}^\varepsilon, & A_{t',\tau}^\varepsilon &= \sigma_z \int_{-\infty}^{t'} \mathcal{K}^\varepsilon(\tau-u) dW_u, \\
B_{t',t,\tau}^\varepsilon &= \sigma_z \int_{t'}^t \mathcal{K}^\varepsilon(\tau-u) dW_u, & C_{t,\tau}^\varepsilon &= \sigma_z \int_t^\tau \mathcal{K}^\varepsilon(\tau-u) dW_u,
\end{aligned}$$

with $A_{t',\tau}^\varepsilon, B_{t',t,\tau}^\varepsilon, C_{t,\tau}^\varepsilon$ being independent and in particular $A_{t',\tau}^\varepsilon$ is $\mathcal{F}_{t'}$ adapted. Therefore, with $s' \geq s \geq t > t'$, we have

$$\begin{aligned}
&\mathbb{E} \left[\left(\mathbb{E} \left[\mathbb{E}[G'(Z_s^\varepsilon) | \mathcal{F}_t] \mathbb{E}[G'(Z_{s'}^\varepsilon) | \mathcal{F}_t] \middle| \mathcal{F}_{t'} \right] - \mathbb{E} \left[\mathbb{E}[G'(Z_s^\varepsilon) | \mathcal{F}_t] \mathbb{E}[G'(Z_{s'}^\varepsilon) | \mathcal{F}_t] \right] \right)^2 \right] \\
&= \mathbb{E} \left[\mathbb{E} \left[\mathbb{E}[G'(Z_s^\varepsilon) | \mathcal{F}_t] \mathbb{E}[G'(Z_{s'}^\varepsilon) | \mathcal{F}_t] \middle| \mathcal{F}_{t'} \right]^2 \right] - \mathbb{E} \left[\mathbb{E}[G'(Z_s^\varepsilon) | \mathcal{F}_t] \mathbb{E}[G'(Z_{s'}^\varepsilon) | \mathcal{F}_t] \right]^2 \\
&= \mathbb{E} \left[G'(A_{t',s}^\varepsilon + B_{t',t,s}^\varepsilon + C_{t,s}^\varepsilon) G'(A_{t',s'}^\varepsilon + B_{t',t,s'}^\varepsilon + \tilde{C}_{t,s'}^\varepsilon) G'(A_{t',s}^\varepsilon + \tilde{B}_{t',t,s}^\varepsilon + \tilde{\tilde{C}}_{t,s}^\varepsilon) \right. \\
&\quad \times G'(A_{t',s'}^\varepsilon + \tilde{B}_{t',t,s'}^\varepsilon + \tilde{\tilde{C}}_{t,s'}^\varepsilon) - G'(A_{t',s}^\varepsilon + B_{t',t,s}^\varepsilon + C_{t,s}^\varepsilon) G'(A_{t',s'}^\varepsilon + B_{t',t,s'}^\varepsilon + \tilde{C}_{t,s'}^\varepsilon) \\
&\quad \left. \times G'(\tilde{A}_{t',s}^\varepsilon + \tilde{B}_{t',t,s}^\varepsilon + \tilde{\tilde{C}}_{t,s}^\varepsilon) G'(\tilde{A}_{t',s'}^\varepsilon + \tilde{B}_{t',t,s'}^\varepsilon + \tilde{\tilde{C}}_{t,s'}^\varepsilon) \right],
\end{aligned}$$

where each additional “tilde” refers to a new independent copy of $A_{t',s}^\varepsilon, B_{t',t,s}^\varepsilon, C_{t,s}^\varepsilon$.

We can then write

$$\begin{aligned}
& \mathbb{E} \left[\left(\mathbb{E} \left[\mathbb{E} [G'(Z_s^\varepsilon) | \mathcal{F}_t] \mathbb{E} [G'(Z_{s'}^\varepsilon) | \mathcal{F}_t] | \mathcal{F}_{t'} \right] - \mathbb{E} \left[\mathbb{E} [G'(Z_s^\varepsilon) | \mathcal{F}_t] \mathbb{E} [G'(Z_{s'}^\varepsilon) | \mathcal{F}_t] \right] \right)^2 \right] \\
& \leq \|G'\|_\infty^2 \mathbb{E} \left[\left(G'(A_{t',s}^\varepsilon + \tilde{B}_{t',t,s}^\varepsilon + \tilde{C}_{t,s}^\varepsilon) G'(A_{t',s'}^\varepsilon + \tilde{B}_{t',t,s'}^\varepsilon + \tilde{C}_{t,s'}^\varepsilon) \right. \right. \\
& \quad \left. \left. - G'(\tilde{A}_{t',s}^\varepsilon + \tilde{B}_{t',t,s}^\varepsilon + \tilde{C}_{t,s}^\varepsilon) G'(\tilde{A}_{t',s'}^\varepsilon + \tilde{B}_{t',t,s'}^\varepsilon + \tilde{C}_{t,s'}^\varepsilon) \right)^2 \right]^{1/2} \\
& \leq 2 \|G'\|_\infty^2 \|G'G''\|_\infty \left(\mathbb{E} [(A_{t',s}^\varepsilon - \tilde{A}_{t',s}^\varepsilon)^2]^{1/2} + \mathbb{E} [(A_{t',s'}^\varepsilon - \tilde{A}_{t',s'}^\varepsilon)^2]^{1/2} \right) \\
& \leq 2\sqrt{2} \|G'\|_\infty^2 \|G'G''\|_\infty \left(\mathbb{E} [(A_{t',s}^\varepsilon)^2]^{1/2} + \mathbb{E} [(A_{t',s'}^\varepsilon)^2]^{1/2} \right) \\
& \leq 2\sqrt{2} \|G'\|_\infty^2 \|G'G''\|_\infty \left[\left(\sigma_z^2 \int_{-\infty}^{t'} \mathcal{K}^\varepsilon(s-u)^2 du \right)^{1/2} + \left(\sigma_z^2 \int_{-\infty}^{t'} \mathcal{K}^\varepsilon(s'-u)^2 du \right)^{1/2} \right] \\
& \leq 4\sqrt{2} \|G'\|_\infty^2 \|G'G''\|_\infty \sigma_{t'-t, \infty}^\varepsilon \leq C_1 (1 \wedge (\varepsilon/(t'-t))^{d-\frac{1}{2}}),
\end{aligned}$$

where we used Lemma A.10 in the last inequality. Then, using the fact that $\mathcal{K} \in L^1$, this gives

$$\begin{aligned}
|\text{Cov}((\vartheta_t^\varepsilon)^2, (\vartheta_{t'}^\varepsilon)^2)| & \leq C_2 \left(\int_t^T ds |\mathcal{K}^\varepsilon(s-t)| \int_{t'}^T du |\mathcal{K}^\varepsilon(u-t)| \right)^2 (1 \wedge (\varepsilon/(t'-t)))^{\frac{d}{2}-\frac{1}{4}} \\
& \leq C_3 \varepsilon^2 (1 \wedge (\varepsilon/(t'-t)))^{\frac{d}{2}-\frac{1}{4}},
\end{aligned}$$

from which the lemma follows. \square

LEMMA A.10. Let $\sigma_{t,\infty}^\varepsilon$ be defined by (A.5). Then there exists $C > 0$ such that

$$\sigma_{t,\infty}^\varepsilon \leq C (1 \wedge (\varepsilon/t)^{d-\frac{1}{2}}). \quad (\text{A.15})$$

Proof. By assumption there exists $K, t_0 > 0$ so that $|\mathcal{K}(t)| \leq Kt^{-d}$ for $t \geq t_0$ with $d > 1$. Therefore, for $t \geq \varepsilon t_0$:

$$\int_t^\infty \mathcal{K}^\varepsilon(s)^2 ds \leq \varepsilon^{2d-1} \int_t^\infty K^2 s^{-2d} ds = \frac{K^2}{2d-1} \left(\frac{\varepsilon}{t}\right)^{2d-1}.$$

For $t < \varepsilon t_0$ we have $\int_t^\infty \mathcal{K}^\varepsilon(s)^2 ds \leq 1$ since $\mathcal{K} \in L^2(0, \infty)$ with a L^2 -norm equal to one. This gives the desired result. \square

Let \tilde{X}_t be defined by (5.36). Then we have

$$Q_t^{(0)}(\tilde{X}_t) = \mathbb{E} \left[h(\tilde{X}_T) \mid \mathcal{F}_t \right]. \quad (\text{A.16})$$

We finish this appendix with three *effective market lemmas*:

LEMMA A.11. Let $f(t, x)$ be smooth bounded and with bounded derivatives. Let X_t be defined by Eq. (3.1) and \tilde{X}_t be defined by Eq. (5.36). For $t, t' \in [0, T]$ we have

$$\sup_{\varepsilon \in (0,1]} \varepsilon^{-1/2} \mathbb{E} \left[\left| \mathbb{E}[f(t, X_t) - f(t, \tilde{X}_t) \mid \mathcal{F}_0] \right|^2 \right]^{1/2} < \infty, \quad (\text{A.17})$$

$$\sup_{\varepsilon \in (0,1]} \varepsilon^{-1/2} \mathbb{E} \left[\left| \mathbb{E}[f(t, X_t)f(t', X_{t'}) - f(t, \tilde{X}_t)f(t', \tilde{X}_{t'}) \mid \mathcal{F}_0] \right|^2 \right]^{1/2} < \infty. \quad (\text{A.18})$$

Proof. The first result follows from Proposition 4.1: take $h(x) = f(t, x)$, $T = t$, and look at $M_0 = \mathbb{E}[h(X_t) \mid \mathcal{F}_0] = \mathbb{E}[f(t, X_t) \mid \mathcal{F}_0]$. Then Proposition 4.1 states that $\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1/2} \mathbb{E}[|M_0 - Q_0^\varepsilon(X_0)|^2]^{1/2} = 0$ with $Q_0^\varepsilon(x) = Q_0^{(0)}(x) + \varepsilon^{1/2} \rho Q_0^{(1)}(x)$ and $Q_0^{(0)}(X_0) = \mathbb{E}[h(\tilde{X}_t) \mid \mathcal{F}_0] = \mathbb{E}[f(t, \tilde{X}_t) \mid \mathcal{F}_0]$. This gives (A.17). The second result follows from the first part noting that $(ab - \tilde{a}\tilde{b}) = b(a - \tilde{a}) + \tilde{a}(b - \tilde{b})$. \square

LEMMA A.12. *Let $f(t, x)$ be smooth bounded and with bounded derivatives. Then we have for $0 \leq t \leq T$:*

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\left(\int_0^t f(s, X_s) ((\sigma_s^\varepsilon)^2 - \bar{\sigma}^2) ds \right)^2 \right] = 0. \quad (\text{A.19})$$

Proof. This follows via an argument as in the proof of Eq. (5.28) for $j = 2$ as given in [13]. \square

LEMMA A.13. *Let $f_j(t, x)$, $j = 1, 2$ be smooth bounded functions and with bounded derivatives and satisfying:*

$$\mathcal{L}_{\text{BS}}(\bar{\sigma}) f_j(t, x) = 0, \quad j = 1, 2. \quad (\text{A.20})$$

Let X_t be defined by Eq. (3.1) and \tilde{X}_t be defined by Eq. (5.36). Then we have for $t \in (0, T)$:

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\left| \mathbb{E} \left[\left| \int_0^t f_1(s, X_s) ds \right|^2 \mid \mathcal{F}_0 \right] - 2 \mathbb{E} \left[\int_0^t f_1(s, \tilde{X}_s)^2 (t-s) ds \mid \mathcal{F}_0 \right] \right| \right] = 0, \quad (\text{A.21})$$

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\left| \varepsilon^{-1/2} \mathbb{E} \left[\int_0^t f_1(s, X_s) ds \int_0^t f_2(s, X_s) d\psi_s^\varepsilon \mid \mathcal{F}_0 \right] \right. \right. \\ & \quad \left. \left. - \rho \bar{D} \mathbb{E} \left[\int_0^t (t-s) \left((x \partial_x) f_1(s, \tilde{X}_s) \right) f_2(s, \tilde{X}_s) ds \mid \mathcal{F}_0 \right] \right| \right] = 0, \quad (\text{A.22}) \end{aligned}$$

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\left| \varepsilon^{-1} \mathbb{E} \left[\left(\int_0^t f_1(s, X_s) d\psi_s^\varepsilon \right)^2 \mid \mathcal{F}_0 \right] - \bar{\Gamma}^2 \mathbb{E} \left[\int_0^t f_1(s, \tilde{X}_s)^2 ds \mid \mathcal{F}_0 \right] \right| \right] = 0. \quad (\text{A.23})$$

Proof. Proof of (A.21): Note first that in view of Lemma A.11, Eq. (A.18), we have

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\left| \mathbb{E} \left[\left| \int_0^t f_1(s, X_s) ds \right|^2 \mid \mathcal{F}_0 \right] - \mathbb{E} \left[\left| \int_0^t f_1(s, \tilde{X}_s) ds \right|^2 \mid \mathcal{F}_0 \right] \right| \right] = 0.$$

Note next that in view of Eq. (A.20) $f_1(s, \tilde{X}_s)$ is a martingale so that

$$\begin{aligned} & \mathbb{E} \left[\left| \int_0^t f_1(s, \tilde{X}_s) ds \right|^2 \mid \mathcal{F}_0 \right] = \mathbb{E} \left[2 \int_0^t f_1(s, \tilde{X}_s) \int_s^t f_1(u, \tilde{X}_u) du ds \mid \mathcal{F}_0 \right] \\ & = \mathbb{E} \left[2 \int_0^t f_1(s, \tilde{X}_s)^2 (t-s) ds \mid \mathcal{F}_0 \right], \end{aligned}$$

which gives Eq. (A.21).

Proof of (A.22): It follows from the fact that $\int_0^t f_2(u, X_u) d\psi_u^\varepsilon$ is a martingale and Itô's Lemma that

$$\begin{aligned}
& \mathbb{E} \left[\int_0^t f_1(s, X_s) ds \int_0^t f_2(s, X_s) d\psi_s^\varepsilon \mid \mathcal{F}_0 \right] \\
&= \int_0^t \mathbb{E} \left[\mathbb{E} \left[\int_0^t f_2(u, X_u) d\psi_u^\varepsilon \mid \mathcal{F}_s \right] f_1(s, X_s) \mid \mathcal{F}_0 \right] ds \\
&= \int_0^t \mathbb{E} \left[\int_0^s f_2(u, X_u) d\psi_u^\varepsilon f_1(s, X_s) \mid \mathcal{F}_0 \right] ds \\
&= \int_0^t \mathbb{E} \left[\int_0^s f_2(u, X_u) d\psi_u^\varepsilon \int_0^s (x^2 \partial_x^2) f_1(u, X_u) \frac{1}{2} ((\sigma_u^\varepsilon)^2 - \bar{\sigma}^2) du \mid \mathcal{F}_0 \right] ds \\
&\quad + \int_0^t \mathbb{E} \left[\int_0^s f_2(u, X_u) d\psi_u^\varepsilon \int_0^s (x \partial_x) f_1(u, X_u) \sigma_u^\varepsilon dW_u^* \mid \mathcal{F}_0 \right] ds \\
&\quad + \int_0^t \mathbb{E} \left[\int_0^s f_2(u, X_u) d\psi_u^\varepsilon f_1(0, X_0) \mid \mathcal{F}_0 \right] ds. \tag{A.24}
\end{aligned}$$

The last term of Eq. (A.24) is zero because $\int_0^t f_2(u, X_u) d\psi_u^\varepsilon$ is a zero-mean martingale. It follows from Lemmas A.5 and A.12 that

$$\begin{aligned}
& \mathbb{E} \left[\left| \mathbb{E} \left[\varepsilon^{-1/2} \int_0^s f_2(u, X_u) d\psi_u^\varepsilon \int_0^s (x^2 \partial_x^2) f_1(u, X_u) \frac{1}{2} ((\sigma_u^\varepsilon)^2 - \bar{\sigma}^2) du \mid \mathcal{F}_0 \right] \right| \right] \\
&\leq \mathbb{E} \left[\varepsilon^{-1} \left(\int_0^s f_2(u, X_u) d\psi_u^\varepsilon \right)^2 \right]^{1/2} \mathbb{E} \left[\left(\int_0^s (x^2 \partial_x^2) f_1(u, X_u) \frac{1}{2} ((\sigma_u^\varepsilon)^2 - \bar{\sigma}^2) du \right)^2 \right]^{1/2} \\
&= \mathbb{E} \left[\varepsilon^{-1} \int_0^s f_2(u, X_u) (\vartheta_u^\varepsilon)^2 du \right]^{1/2} \mathbb{E} \left[\left(\int_0^s (x^2 \partial_x^2) f_1(u, X_u) \frac{1}{2} ((\sigma_u^\varepsilon)^2 - \bar{\sigma}^2) du \right)^2 \right]^{1/2} \\
&\xrightarrow{\varepsilon \rightarrow 0} 0. \tag{A.25}
\end{aligned}$$

By Lemma A.4 we have

$$\begin{aligned}
& \mathbb{E} \left[\varepsilon^{-1/2} \int_0^s f_2(u, X_u) d\psi_u^\varepsilon \int_0^s (x \partial_x) f_1(u, X_u) \sigma_u^\varepsilon dW_u^* \mid \mathcal{F}_0 \right] \\
&\quad - \rho \mathbb{E} \left[\int_0^s f_2(u, X_u) (x \partial_x) f_1(u, X_u) \bar{D} du \mid \mathcal{F}_0 \right] \\
&= \rho \mathbb{E} \left[\int_0^s f_2(u, X_u) (x \partial_x) f_1(u, X_u) (\varepsilon^{-1/2} \sigma_u^\varepsilon \vartheta_u^\varepsilon - \bar{D}) du \mid \mathcal{F}_0 \right].
\end{aligned}$$

By Lemma A.6, Eq. (A.13), we get

$$\begin{aligned}
& \mathbb{E} \left[\left| \mathbb{E} \left[\int_0^s f_2(u, X_u) (x \partial_x) f_1(u, X_u) (\varepsilon^{-1/2} \sigma_u^\varepsilon \vartheta_u^\varepsilon - \bar{D}) du \mid \mathcal{F}_0 \right] \right| \right] \\
&\leq \mathbb{E} \left[\left(\int_0^s f_2(u, X_u) (x \partial_x) f_1(u, X_u) (\varepsilon^{-1/2} \sigma_u^\varepsilon \vartheta_u^\varepsilon - \bar{D}) du \right)^2 \right]^{1/2} \xrightarrow{\varepsilon \rightarrow 0} 0.
\end{aligned}$$

By Lemma A.11, Eq. (A.17), we find

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\left| \mathbb{E} \left[\int_0^s f_2(u, X_u) (x \partial_x) f_1(u, X_u) du \mid \mathcal{F}_0 \right] \right. \right. \\
&\quad \left. \left. - \mathbb{E} \left[\int_0^s f_2(u, \tilde{X}_u) (x \partial_x) f_1(u, \tilde{X}_u) du \mid \mathcal{F}_0 \right] \right| \right] = 0.
\end{aligned}$$

Therefore

$$\mathbb{E} \left[\left| \mathbb{E} \left[\varepsilon^{-1/2} \int_0^s f_2(u, X_u) d\psi_u^\varepsilon \int_0^s (x\partial_x) f_1(u, X_u) \sigma_u^\varepsilon dW_u^* \mid \mathcal{F}_0 \right] - \rho \bar{D} \mathbb{E} \left[\int_0^s f_2(u, \tilde{X}_u) (x\partial_x) f_1(u, \tilde{X}_u) du \mid \mathcal{F}_0 \right] \right| \right] = 0. \quad (\text{A.26})$$

We substitute (A.25-A.26) into (A.24) and we invoke Lebesgue's dominated convergence theorem (because $\varepsilon^{-1/2} \vartheta_u^\varepsilon$ is uniformly bounded) to prove (A.22).

Proof of (A.23): The result follows from Lemmas A.4, A.6 and A.11. \square

Appendix B. The fOU Volatility Factor.

We use a rapid fractional Ornstein-Uhlenbeck (fOU) process as the volatility factor and describe here how this process can be represented in terms of a fractional Brownian motion. Since fractional Brownian motion can be expressed in terms of ordinary Brownian motion we also arrive at an expression for the rapid fOU process as a filtered version of Brownian motion.

A fractional Brownian motion (fBM) is a zero-mean Gaussian process $(W_t^H)_{t \in \mathbb{R}}$ with the covariance

$$\mathbb{E}[W_t^H W_s^H] = \frac{\sigma_H^2}{2} (|t|^{2H} + |s|^{2H} - |t-s|^{2H}), \quad (\text{B.1})$$

where σ_H is a positive constant. We use the following moving-average stochastic integral representation of the fBM [18]:

$$W_t^H = \frac{1}{\Gamma(H + \frac{1}{2})} \int_{\mathbb{R}} (t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}} dW_s, \quad (\text{B.2})$$

where $(W_t)_{t \in \mathbb{R}}$ is a standard Brownian motion over \mathbb{R} . Then indeed $(W_t^H)_{t \in \mathbb{R}}$ is a zero-mean Gaussian process with the covariance (B.1) and we have

$$\sigma_H^2 = \frac{1}{\Gamma(2H+1) \sin(\pi H)}. \quad (\text{B.3})$$

We introduce the ε -scaled fractional Ornstein-Uhlenbeck process (fOU) as

$$Z_t^\varepsilon = \sqrt{2 \sin(\pi H)} \sigma_Z \varepsilon^{-H} \int_{-\infty}^t e^{-\frac{t-s}{\varepsilon}} dW_s^H. \quad (\text{B.4})$$

The fractional OU process can be seen as a fractional Brownian motion with a restoring force towards zero. It is a zero-mean, stationary Gaussian process, with variance

$$\mathbb{E}[(Z_t^\varepsilon)^2] = \sigma_Z^2, \quad (\text{B.5})$$

that is independent of ε , and covariance:

$$\mathbb{E}[Z_t^\varepsilon Z_{t+s}^\varepsilon] = \sigma_Z^2 \mathcal{C}_Z \left(\frac{s}{\varepsilon} \right), \quad (\text{B.6})$$

that is a function of s/ε only, with

$$\begin{aligned} \mathcal{C}_Z(s) &= \frac{1}{\Gamma(2H+1)} \left[\frac{1}{2} \int_{\mathbb{R}} e^{-|v||s+v|^{2H}} dv - |s|^{2H} \right] \\ &= \frac{2 \sin(\pi H)}{\pi} \int_0^\infty \cos(sx) \frac{x^{1-2H}}{1+x^2} dx. \end{aligned} \quad (\text{B.7})$$

This shows that ε is the natural scale of variation of the fOU Z_t^ε . Note that the random process Z_t^ε is not a martingale, neither a Markov process. For $H \in (0, 1/2)$ it possesses short-range correlation properties in the sense that its correlation function is rough at zero as seen in (3.9) while it is integrable and it decays as s^{2H-2} at infinity as seen in (3.10).

Using Eqs. (B.2) and (B.4) we arrive at the moving-average integral representation of the scaled fOU as:

$$Z_t^\varepsilon = \sigma_z \int_{-\infty}^t \mathcal{K}^\varepsilon(t-s) dW_s, \quad (\text{B.8})$$

where \mathcal{K}^ε is of the form (3.4)-(3.8). The kernel \mathcal{K} satisfies the assumptions set forth in Section 3 and the main properties are the following ones (valid for any $H \in (0, 1/2)$):

- (i) $\mathcal{K} \in L^2(0, \infty)$ with $\int_0^\infty \mathcal{K}^2(u) du = 1$ and $\mathcal{K} \in L^1(0, \infty)$.
- (ii) For small times $t \ll 1$:

$$\mathcal{K}(t) = \frac{\sqrt{2 \sin(\pi H)}}{\Gamma(H + \frac{1}{2})} \left(t^{H-\frac{1}{2}} + O(t^{H+\frac{1}{2}}) \right). \quad (\text{B.9})$$

- (iii) For large times $t \gg 1$:

$$\mathcal{K}(t) = \frac{\sqrt{2 \sin(\pi H)}}{\Gamma(H - \frac{1}{2})} \left(t^{H-\frac{3}{2}} + O(t^{H-\frac{5}{2}}) \right). \quad (\text{B.10})$$

The volatility process σ_t^ε defined by (3.2) inherits the short-range correlation properties of the volatility driving process Z_t^ε . This follows from the following lemma proved in [13]:

LEMMA B.1. *We denote, for $j = 1, 2$:*

$$\langle F^j \rangle = \int_{\mathbb{R}} F(\sigma_z z)^j p(z) dz, \quad \langle F'^j \rangle = \int_{\mathbb{R}} F'(\sigma_z z)^j p(z) dz, \quad (\text{B.11})$$

where $p(z)$ is the pdf of the standard normal distribution.

1. The process σ_t^ε is a stationary random process with mean $\mathbb{E}[\sigma_t^\varepsilon] = \langle F \rangle$ and variance $\text{Var}(\sigma_t^\varepsilon) = \langle F^2 \rangle - \langle F \rangle^2$, independently of ε .
2. The covariance function of the process σ_t^ε is of the form

$$\text{Cov}(\sigma_t^\varepsilon, \sigma_{t+s}^\varepsilon) = (\langle F^2 \rangle - \langle F \rangle^2) \mathcal{C}_\sigma\left(\frac{s}{\varepsilon}\right), \quad (\text{B.12})$$

where the correlation function \mathcal{C}_σ satisfies $\mathcal{C}_\sigma(0) = 1$ and

$$\mathcal{C}_\sigma(s) = 1 - \frac{1}{\Gamma(2H+1)} \frac{\sigma_z^2 \langle F'^2 \rangle}{\langle F^2 \rangle - \langle F \rangle^2} s^{2H} + o(s^{2H}), \quad \text{for } s \ll 1, \quad (\text{B.13})$$

$$\mathcal{C}_\sigma(s) = \frac{1}{\Gamma(2H-1)} \frac{\sigma_z^2 \langle F' \rangle^2}{\langle F^2 \rangle - \langle F \rangle^2} s^{2H-2} + o(s^{2H-2}), \quad \text{for } s \gg 1. \quad (\text{B.14})$$

Consequently, the process σ_t^ε has short-range correlation properties and its covariance function is integrable.

Appendix C. Call Option Hedging Cost and Risk.

We use below the following ‘‘Greek’’ identities for the European call case:

$$\frac{x^2 \partial_x^2 Q_t^{(0)}(x)}{K} = \frac{\partial_\sigma Q_t^{(0)}(x)}{K \bar{\sigma}(T-t)} = \frac{(x/K) e^{-d_+^2(x,t)/2}}{\sqrt{2\pi\tau_t}} = \frac{e^{-d_-^2(x,t)/2}}{\sqrt{2\pi\tau_t}}, \quad (\text{C.1})$$

$$\frac{x \partial_x x^2 \partial_x^2 Q_t^{(0)}(x)}{K} = -\frac{d_-(x,t) e^{-d_-^2(x,t)/2}}{\sqrt{2\pi\tau_t}}, \quad (\text{C.2})$$

$$\frac{(x \partial_x)^2 x^2 \partial_x^2 Q_t^{(0)}(x)}{K} = \frac{(d_-^2(x,t) - 1) e^{-d_-^2(x,t)/2}}{\sqrt{2\pi\tau_t}^{3/2}}, \quad (\text{C.3})$$

with $\tau_t = (T-t)\bar{\sigma}^2$ and

$$d_\pm(x,t) = \frac{\log(x/K)}{\sqrt{\tau_t}} \pm \frac{\sqrt{\tau_t}}{2}.$$

We will also use the following lemma:

LEMMA C.1.

$$\mathbb{E} \left[e^{-d_-^2(\tilde{X}_{Ts}, Ts)} \mid \mathcal{F}_0 \right] = \exp \left(-\frac{d_-^2(X_0, 0)}{1+s} \right) f_0(s), \quad (\text{C.4})$$

$$f_0(s) = \sqrt{\frac{1-s}{1+s}}, \quad (\text{C.5})$$

$$\mathbb{E} \left[d_-^2(\tilde{X}_{Ts}, Ts) e^{-d_-^2(\tilde{X}_{Ts}, Ts)} \mid \mathcal{F}_0 \right] = \exp \left(-\frac{d_-^2(X_0, 0)}{1+s} \right) f_2(s, d_-(X_0, 0)), \quad (\text{C.6})$$

$$f_2(s, d) = d^2 \sqrt{\frac{(1-s)^3}{(1+s)^5}} + s \sqrt{\frac{1-s}{(1+s)^3}}, \quad (\text{C.7})$$

$$\mathbb{E} \left[d_-^4(\tilde{X}_{Ts}, Ts) e^{-d_-^2(\tilde{X}_{Ts}, Ts)} \mid \mathcal{F}_0 \right] = \exp \left(-\frac{d_-^2(X_0, 0)}{1+s} \right) f_4(s, d_-(X_0, 0)), \quad (\text{C.8})$$

$$f_4(s, d) = d^4 \sqrt{\frac{(1-s)^5}{(1+s)^9}} + 6d^2 s \sqrt{\frac{(1-s)^3}{(1+s)^7}} + 3s^2 \sqrt{\frac{1-s}{(1+s)^5}}. \quad (\text{C.9})$$

Proof. By (5.36), for any t , we have in distribution

$$\tilde{X}_t = X_0 \exp \left(\bar{\sigma} W_t^* - \frac{1}{2} \bar{\sigma}^2 t \right) = X_0 \exp \left(\bar{\sigma} \sqrt{t} Z - \frac{1}{2} \bar{\sigma}^2 t \right),$$

with Z having the standard normal distribution, and therefore

$$d_-(\tilde{X}_t, t) = \frac{\log(\tilde{X}_t/K)}{\sqrt{\tau_t}} - \frac{\sqrt{\tau_t}}{2} = \frac{d_-(X_0, 0) + Z \sqrt{t/T}}{\sqrt{1-t/T}}.$$

One can then carry out the resulting Gaussian integral upon a completion of the square in the exponential. \square

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